Section 2 – 3: Proportions

A proportion is an equation that states that two ratios or rates each written as fractions are equal to each other. A proportion is made by writing an equation with two proper or improper fractions set equal to each other. We do not use mixed numbers when we write a proportion. It is common for the fractions in the proportion to have numbers that have units of measure. If that happens then the units of measure must be included when you write the proportion.

\[
\frac{3}{4} = \frac{6}{8} \quad \frac{3}{8} = \frac{x}{24} \quad \frac{2}{5} = \frac{6}{15} \quad \frac{30}{2} = \frac{120}{x}
\]

The Cross Product Rule for Proportions

The Cross Product Rule for Proportions states that the two cross products can be set equal to each other to form a new equation.

The Cross Product Rule for Proportions

Given the Proportion:

\[
\frac{A}{B} = \frac{C}{D}
\]

\[
\frac{A}{B} \times \frac{C}{D}
\]

The cross products are

\[
A \cdot D \quad \text{and} \quad B \cdot C
\]

and

\[
A \cdot D = B \cdot C
\]

Solving Proportions with a Variable Term

If one of the terms in a proportion is an unknown variable then we can find the value for that variable term by using the Cross Product Rule for Proportions. Use the Cross Product Rule to create a simple one step equation and then solve that equation for x.

The Cross Product Rule for Proportions

Given the Proportion:

\[
\frac{2}{5} = \frac{x}{10}
\]

The cross products are

\[
2 \cdot 10 \quad \text{and} \quad 5 \cdot x
\]

and

\[
20 = 5x
\]

The Cross Product Rule for Proportions

Given the Proportion:

\[
\frac{x}{3} = \frac{10}{15}
\]

The cross products are

\[
x \cdot 15 \quad \text{and} \quad 3 \cdot 10
\]

and

\[
15x = 30
\]
Solving a Proportion for an Unknown

If one of the numbers in a proportion is unknown then we can find the value for that number by using the **Cross Product Rule** for proportions.

1. Set the cross products equal to each other.
2. Solve the equation for \( x \).
3. Check your answer.

### Example 1

Solve the Proportion
\[
\frac{3}{4} = \frac{x}{12}
\]

The cross products are
\[
4 \times x \quad \text{and} \quad 3 \times 12
\]

\[
4 \times x = 3 \times 12
\]

\[
x = 9
\]

### Example 2

Solve the Proportion
\[
\frac{3}{2} = \frac{15}{x}
\]

The cross products are
\[
3 \times x \quad \text{and} \quad 2 \times 15
\]

\[
3 \times x = 2 \times 15
\]

\[
x = 10
\]

### Example 3

Solve the Proportion
\[
\frac{5}{2} = \frac{x}{7}
\]

cross multiply and solve for \( x \)

\[
2 \times x = 5 \times 7
\]

\[
2x = 35
\]

divide both sides by 2

\[
x = \frac{35}{2} = 17.5
\]

### Example 4

Solve the Proportion
\[
\frac{7}{3} = \frac{x}{12}
\]

cross multiply and solve for \( x \)

\[
2 \times x = \frac{7}{3} \times 12
\]

\[
2x = 28
\]

divide both sides by 2

\[
x = 14
\]
Solving Proportions Word Problems Involving Rates

There are many real world problems that involve proportions. The most common ones involve rates. A rate is a ratio of two numbers each with a different unit. The first rate is compared to a second rate but one of the numbers is unknown and its value must be found. To solve these problems we set up a proportion and then solve for the unknown. It is important to keep the units attached to the number values. You must also be sure that the numerators of the proportion have numbers with a common unit and the denominators also have numbers with a common unit.

Example 1: A hospital requires a staffing ratio of three nurses for every two doctors. If the hospital employs 24 doctors, how many nurses are employed?

Step 1. Find the sentence that compares two given values as a ratio. Write that ratio as a fraction and make sure that the units are attached to the values.

\[
\frac{3 \text{ nurses}}{2 \text{ doctors}}
\]

Step 2. Find the sentence that asks you to find a missing value given a known value. Write a fraction with the known and unknown with the units they come with. Make sure that this fraction has the units in the same position as the first fraction.

\[
\frac{x \text{ nurses}}{24 \text{ doctors}}
\]

Step 3. Set the fraction in Step 1 equal to the fraction in Step 2.

\[
\frac{3 \text{ nurses}}{2 \text{ doctors}} = \frac{x \text{ nurses}}{24 \text{ doctors}}
\]

Step 4. Set the cross products equal to each other and solve for the unknown value. Be sure to include units in the answer.

The cross products are

\[
2 \cdot x = 3 \cdot 24
\]

\[
x = 36 \text{ nurses}
\]

Step 5. Check your answer.

Check:

\[
\frac{36 \text{ nurses}}{24 \text{ doctors}} = \frac{36^3}{2^2} = \frac{3 \text{ nurses}}{2 \text{ doctors}}
\]
Example 2: An engine requires a ratio of 5 ounces of oil for every 6 gallons of gas. If we need 30 gallons of gas how much oil should we add?

The ratio of oil to gas is 5 ounces to 6 gallons

\[
\frac{5 \text{ ounces}}{6 \text{ gallons}}
\]

We have 30 gallons for an unknown number of ounces

\[
\frac{5 \text{ ounces}}{6 \text{ gallons}} = \frac{x \text{ ounces}}{30 \text{ gallons}}
\]

cross multiply and solve for x

\[
6 \cdot x = 5 \cdot 30
\]

\[
6x = 150
\]

\[
x = 25 \text{ ounces}
\]

Check:

\[
\frac{25 \text{ ounces}}{30 \text{ gallons}} = \frac{25}{30} = \frac{5}{6} = \frac{5 \text{ ounces}}{6 \text{ gallons}}
\]

Example 3: A 10 foot tree has a 4 foot shadow. How long a shadow will a 25 foot tree have?

The ratio of tree to shadow is 10 feet to 4 feet

\[
\frac{10 \text{ foot tree}}{4 \text{ foot shadow}}
\]

We have a 25 foot tree with an unknown shadow

\[
\frac{10 \text{ foot tree}}{4 \text{ foot shadow}} = \frac{25 \text{ foot tree}}{x \text{ foot shadow}}
\]

cross multiply and solve for x

\[
10 \cdot x = 4 \cdot 25
\]

\[
10x = 100
\]

\[
x = 10 \text{ foot shadow}
\]
Example 4: The scale on a map states that 4 inches on the map is equal to 3 miles in the real world. If the distance on a map is 10 inches apart how many miles are the actual cities apart?

**Does it matter which of the two values goes on the top of the ratio?**

When we say that the ratio of 4 inches is equal to 3 miles we mean that 4 inches on a map equals 3 miles on land. We can compare **inches to miles** or **miles to inches** and the answer will still be the same as long as we keep the matching units in the numerator and matching units in the denominator.

\[ \frac{4 \text{ inches}}{3 \text{ miles}} \]

can be written

\[ \frac{4 \text{ inches}}{3 \text{ miles}} = \frac{x \text{ inches}}{y \text{ miles}} \quad \text{or} \quad \frac{3 \text{ miles}}{4 \text{ inches}} = \frac{x \text{ miles}}{y \text{ inches}} \]

but the fraction we set the ratio equal to

must have the units in

the same place as the first ratio

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**Putting inches over miles**

The ratio of inches to miles is 4 inches to 3 miles

\[ \frac{4 \text{ inches}}{3 \text{ miles}} \]

They are 10 inches apart for an unknown number of miles

\[ \frac{4 \text{ inches}}{3 \text{ miles}} = \frac{10 \text{ inches}}{x \text{ miles}} \]

cross multiply and solve for \( x \)

\[ 4 \cdot x = 3 \cdot 10 \]
\[ 4x = 30 \]
\[ \frac{4x}{4} = \frac{30}{4} \]
\[ x = 7.5 \text{ miles} \]

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**Putting miles over inches**

The ratio of inches to miles is 4 inches to 3 miles

\[ \frac{3 \text{ miles}}{4 \text{ inches}} \]

They are 10 inches apart for an unknown number of miles

\[ \frac{3 \text{ miles}}{4 \text{ inches}} = \frac{x \text{ miles}}{10 \text{ inches}} \]

cross multiply and solve for \( x \)

\[ 4 \cdot x = 3 \cdot 10 \]
\[ 4x = 30 \]
\[ \frac{4x}{4} = \frac{30}{4} \]
\[ x = 7.5 \text{ miles} \]
Example 5: A private college advertises that they have a ratio of one teacher for every 18 full-time students. The college plans to enroll 216 students. How many teachers must be employed?

The ratio of teachers to students is 1 teacher to 18 students

There are 216 students for an unknown number of teachers

\[
\frac{1 \text{ teacher}}{18 \text{ students}} = \frac{x \text{ teachers}}{216 \text{ students}}
\]

cross multiply and solve for \(x\)

\[
18 \cdot x = 1 \cdot 216
\]

\[
18x = 216
\]

\[
\frac{18x}{18} = \frac{216}{18}
\]

\[
x = 12 \text{ teachers}
\]

Check:

\[
\frac{12 \text{ teachers}}{216 \text{ students}} = \frac{12^1}{216^{18}}
\]