The first sections of this chapter will ask questions about where the graph of a function has “high” points and “low” points. We will also be interested in ranges of values of x values where the function is decreasing or increasing as you move from left to right on the graph. This section will give a general overview of the topics without giving a detailed definition for high points, low points, or interval where the y values are increasing or decreasing.

The graph below is a smooth curve with no gaps or breaks.

**Decreasing:** The values for y are large on the left side of the graph $x \to -\infty$ and get smaller and smaller (decreasing) until the point $(a,f(a))$. We will say that the y values of the function are decreasing for the x values on the open interval $-\infty < x < a$.

The values for y get smaller and smaller (decreasing) from the point $(b,f(b))$ until you get to the point $(c,f(c))$. We will say that the y values of the function are decreasing for the x values on the open interval $b < x < c$. The y values of the function are decreasing for x values on the open interval $-\infty < x < a \cup b < x < c$.

**Increasing:** The values for y get larger and larger (increasing) from the point $(a,f(a))$ until you get to the point $(b,f(b))$. We will say that the y values of the function are increasing for the x values on the open interval $a < x < b$. The values for y get larger and larger (increasing) from the point $(c,f(c))$ and continue to increase as $x \to -\infty$. We will say that the y values of the function are increasing for the x values on the open interval $c < x < +\infty$. The y values of the function are increasing for x values on the open intervals $a < x < b \cup c < x < -\infty$. 
Relative Minimum Point: A relative minimum point It is the lowest point in the small local neighborhood around the low point. A relative minimum point is a point where the y values of a function are decreasing to the LEFT of the point and increasing to the right of a point. The graph of the function above has relative minimum points at \((a, f(a))\) and \((c, f(c))\)

Relative Maximum: A relative maximum point It is the highest point in the small local neighborhood around the high point. A relative maximum point is a point where the y values of a function are increasing to the LEFT of the point and decreasing to the right of a point. The graph of the function above has a relative maximum point at \((b, f(b))\)

Absolute Minimum: An absolute minimum point It is the absolute lowest point on the entire graph. Some graphs may not have an absolute minimum but this graph does have one. The absolute lowest point on the entire graph is at \((c, f(c))\)

Absolute Maximum: An absolute maximum point It is the absolute highest point on the entire graph. Some graphs may not have an absolute maximum point because the y values of the points on the graph continue increasing to \(+\infty\). This is the case for this graph so it does not have an absolute maximum point.
Relative Minimum Point: A relative minimum point It is the lowest point in the small local neighborhood around the low point. A relative minimum point is a point where the y values of a function are decreasing to the LEFT of the point and increasing to the right of a point. The graph of the function above has relative minimum points at \((a, f(a))\) and \((c, f(c))\).

Relative Maximum: A relative maximum point It is the highest point in the small local neighborhood around the high point. A relative maximum point is a point where the y values of a function are increasing to the LEFT of the point and decreasing to the right of a point. The graph of the function above has relative maximum points at \((b, f(b))\) and \((d, f(d))\).

Absolute Minimum: An absolute minimum point It is the absolute lowest point on the entire graph. Some graphs may not have an absolute minimum point because the y values of the points on the graph continue decreasing to \(-\infty\). This is the case for this graph so it does not have an absolute minimum point.

Absolute Maximum: An absolute maximum point It is the absolute highest point on the entire graph. Some graphs may not have an absolute maximum point because the y values of the points on the graph continue increasing to \(+\infty\). This is the case for this graph so it does not have an absolute maximum point.
The graph below is a smooth curve with no gaps or breaks.

Relative Minimum Point: A relative minimum point It is the lowest point in the small local neighborhood around the low point. A relative minimum point is a point where the y values of a function are decreasing to the LEFT of the point and increasing to the right of a point. The graph of the function above has relative minimum points at $(b,f(b))$ and $(d,f(d))$ $(c,f(c))$.

Relative Maximum: A relative maximum point It is the highest point in the small local neighborhood around the high point. A relative maximum point is a point where the y values of a function are increasing to the LEFT of the point and decreasing to the right of a point. The graph of the function above has relative maximum points at $(a,f(a))$ and $(c,f(c))$.

Absolute Minimum: An absolute minimum point It is the absolute lowest point on the entire graph. Some graphs may not have an absolute minimum point because the y values of the points on the graph continue decreasing to $-\infty$. This is the case for this graph so it does not have an absolute minimum point.

Absolute Maximum: An absolute maximum point It is the absolute highest point on the entire graph. Some graphs may not have an absolute maximum point because the y values of the points on the graph continue increasing to $+\infty$. This is the case for this graph so it does not have an absolute maximum point.
The graph below is a curve with no gaps or breaks, but it has a sharp point at \((a, f(a))\).

**Relative Minimum Point:** A relative minimum point is the lowest point in the small local neighborhood around the low point. A relative minimum point is a point where the y values of a function are decreasing to the left of the point and increasing to the right of a point. The graph of the function above has relative minimum points at \((b, f(b))\) and \((d, f(d))\).

**Relative Maximum:** A relative maximum point is the highest point in the small local neighborhood around the high point. A relative maximum point is a point where the y values of a function are increasing to the left of the point and decreasing to the right of a point. The graph of the function above has relative maximum points at \((a, f(a))\) and \((c, f(c))\) and \((e, f(e))\).

**Absolute Minimum:** An absolute minimum point is the absolute lowest point on the entire graph. Some graphs may not have an absolute minimum point because the y values of the points on the graph continue decreasing to \(-\infty\). This is the case for this graph so it does not have an absolute minimum point.

**Absolute Maximum:** An absolute maximum point is the absolute highest point on the entire graph. Some graphs may not have an absolute minimum but this graph does have one. The absolute highest point on the entire graph is at \((c, f(c))\).
The graph below is a smooth curve with no gaps or breaks but it has a domain restriction. The x values are restricted to the CLOSED interval \([a, d]\) creating 2 ENDPOINTS \((a, f(a))\) and \((d, f(d))\).

Relative Minimum Point: A relative minimum point It is the lowest point in the small local neighborhood around the low point. A relative minimum point is a point where the y values of a function are decreasing to the LEFT of the point and increasing to the RIGHT of a point. The graph of the function above has a relative minimum points at \((c, f(c))\).

The point \((a, f(a))\) is NOT a relative minimum point because the graph does is not increasing to the left of \((d, f(d))\). We will call this endpoint a endpoint extrema.

Relative Maximum: A relative maximum point It is the highest point in the small local neighborhood around the high point. A relative maximum point is a point where the y values of a function are increasing to the LEFT of the point and decreasing to the RIGHT of a point. The graph of the function above has relative maximum points at \((b, f(b))\).

The point \((d, f(d))\) is NOT a relative maximum point because the graph does is not decreasing to the right of \((d, f(d))\). We will call this endpoint a endpoint extrema.

Absolute Minimum: An absolute minimum point It is the absolute lowest point on the entire graph. Some graphs may not have an absolute minimum point because the y values of the points on the graph continue decreasing to \(-\infty\). This is the case for this graph so it does not have an absolute minimum point. An Endpoint Extrema can be a absolute minimum point.

Absolute Maximum: An absolute maximum point It is the absolute highest point on the entire graph. Some graphs may not have an absolute maximum point because the y values of the points on the graph continue increasing to \(+\infty\). This is the case for this graph so it does not have an absolute maximum point. An Endpoint Extrema can be a absolute maximum point.
Extrema

General Definition of Extrema

Let a function \( f \) be defined on an interval containing \( c \), then:

1. \( f(c) \) is the **maximum** \( y \) value of the function in an interval if \( f(c) \geq f(x) \) for all \( x \) in the interval.

2. \( f(c) \) is the **minimum** \( y \) value of the function in the interval if \( f(c) \leq f(x) \) for all \( x \) in the interval.

The **maximum and minimum** \( y \) values on an interval are called the extreme values or Extrema (plural) of the function on the interval.

Definition of Relative Extrema

(Local Maximum and Local Minimums)

1. Let a function \( f \) be defined on an **OPEN INTERVAL** \( (a,b) \) and let \( x = c \) where \( c \) is in the open interval \( a < c < b \). If \( f(c) \) is the maximum \( y \) value for the function on the **OPEN INTERVAL** \( a < c < b \) then \( f(c) \) is called a **relative maximum** \( y \) value for the function in the interval \( (a,b) \).

   We call \( f(c) \) a **local maximum** \( y \) value because it is **only the maximum** \( y \) value for the open interval \( a < c < b \). A different open interval like \( a_1 < c_1 < b_1 \) may have a different **local maximum** \( y \) value \( f(c_1) \) in the interval \( a_1 < c_1 < b_1 \).

2. Let a function \( f \) be defined on an **OPEN INTERVAL** \( (a,b) \) and let \( x = c \) where \( c \) is in the open interval \( a < c < b \). If \( f(c) \) is the minimum \( y \) value for the function on the **OPEN INTERVAL** \( a < c < b \) then \( f(c) \) is called a **relative minimum** \( y \) value of the function in the interval \( (a,b) \).

   We call \( f(c) \) a **local minimum** \( y \) value because it is **only the minimum value** for the open interval \( (a,b) \). A different open interval like \( a_1 < c_1 < b_1 \) may have a different **local minimum** \( y \) value \( f(c_1) \) in the interval \( a_1 < c_1 < b_1 \).
Endpoints on Closed Intervals

Closed Intervals have endpoints. Each endpoint has a $y$ value that is either the smallest or largest values for all the points on the small closed interval that includes the endpoint. The endpoints are called ENDPOINT EXTREMA.

End point Extrema minimum value or a relative maximum value for the small closed interval that includes the endpoint. These local extrema are called Endpoint Extrema.

If the function $f(x)$ is a continuous function on the closed interval $[a, b]$, then the endpoints $(a, f(a))$ and $(b, f(b))$ are called Endpoint Extrema. The values of $f(a)$ and $f(b)$ are Endpoint Extrema.

Open Intervals have open circles at the end of each graph showing that the endpoint is not included in the interval. Open Intervals CANNOT have endpoint extrema.

The Absolute Maximum Value of a Function and

The Absolute Minimum Value of a Function

1. $f(c)$ is the Absolute Maximum value of the function $f(x)$ if $f(c) \geq f(x)$ for all values of $x$

2. $f(c)$ is the Absolute Minimum value of the function $f(x)$ if $f(c) \leq f(x)$ for all values of $x$

Note:

If the $\lim_{x \to a} F(x) = \infty$ for any value of $a$ in the domain of $f(x)$ then there is NO Absolute Maximum Value for the function.

If the $\lim_{x \to a} F(x) = -\infty$ for any value of $a$ in the domain of $f(x)$ then there is NO Absolute Minimum Value for the function.
Theorem 4.1
The Extreme Value Theorem (Existence)
If \( f(x) \) is a continuous function on the closed interval \([a,b]\) then \( f(x) \) has at least one absolute maximum value and at least one absolute minimum value on the closed interval \([a,b]\). These values occur at an x value on the open interval \((a,b)\) or at an endpoint of the closed interval.

While the Extreme Value Theorem tells us that an absolute maximum and an absolute maximum exist, it does not tell us where to find these points. That problem is solved by the introduction of the and the Critical Numbers of the derivative of the function.

Definition of Critical Number
Given a function \( f(x) \) and a real number \( c \)
If \( f'(c) = 0 \) or if \( f'(c) \) is undefined at \( x=c \)
then \( c \) is a critical number for the function.

Critical Numbers are the value(s) of \( x \) where the derivative of the function equals zero OR where the derivative of the function is undefined.

Theorem 4–2
The Relative Extrema for the function can only occur at critical x values for the function or at the endpoints of closed intervals.

The y value for each critical x number or endpoint of a closed interval is a possible extrema for the function.

If \( x = c \) is a critical number the y value for the point \( (c, f(c)) \) is \( f(c) \)
The value of \( f(c) \) is a possible extrema for the function.

The endpoints of a closed interval \([a,b]\) are \( (a, f(a)) \) and \( (b, f(b)) \)
and the y values for the endpoints are \( f(a) \) and \( f(b) \)
\( f(a) \) and \( f(b) \) also possible extrema for the function.

There are no other possible x values where possible extrema for the function can occur.
The Steps for Finding Relative Extrema on a Closed Interval for a continuous function

If a function $f$ is continuous on a closed interval $[a,b]$ then the following steps will find the Local Extrema:

**Step 1.** Find all critical $x$ values. Find all the values of $x$ where $f'(x) = 0$ or where $f'(x)$ is undefined.

**Step 2.** If the critical $x$ values are $x_1, x_2, ..., x_n$ then the $y$ values for each critical $x$ value are $f(x_1), f(x_2), ..., f(x_n)$ These $y$ values are the possible Relative Extrema.

**Step 3.** If the function is defined on a closed interval $[a,b]$. Find the $y$ values for $f(a)$ and $f(b)$.

**Step 4.** The largest $y$ value from $f(x_1), f(x_2), ..., f(x_n)$, $f(a)$ and $f(b)$ is a maximum value for the function on the closed interval.

The smallest $y$ value from $f(x_1), f(x_2), ..., f(x_n)$, $f(a)$ and $f(b)$ is a minimum value for the function on the closed interval.

**Note:** The function $f$ must be continuous on a closed interval $[a,b]$ for this approach to work. If there is a vertical asymptote in the interval the maximum or minimum values for $y$ will approach $+\infty$ or depending on the function $-\infty$. If the maximum values for $y$ approach $+\infty$ then there will not be an absolute maximum value for $y$. If the maximum values for $y$ approach $+\infty$ then there will not be an absolute minimum value for $y$. 