Derivatives of inverse functions

1. Prove that $\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$.

   Proof: Set $y = \arcsin(x)$ so that we want to show $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$. By the definition of arcsine we know

   $$\boxed{x = \sin(y) \quad (\star)} \qquad \text{and} \qquad \boxed{-\tfrac{\pi}{2} \le y \le \tfrac{\pi}{2} \quad (\star\star)}$$

   Differentiating both sides of $(\star)$ with respect to x gives

   $$\frac{d}{dx}(x) = \frac{d}{dx}(\sin(y)) \quad \Longrightarrow \quad 1 = \frac{d}{dy}(\sin(y))\frac{dy}{dx}$$

   $$\Longrightarrow \quad 1 = \cos(y)\frac{dy}{dx} \quad \Longrightarrow \quad \frac{dy}{dx} = \frac{1}{\cos(y)}.$$

   Now let's use the pythagorean identity:

   $$\sin^2(y)+\cos^2(y) = 1 \quad \Longrightarrow \quad \cos^2(y) = 1-\sin^2(y) \quad \Longrightarrow \quad \cos(y) = \pm\sqrt{1-\sin^2(y)}$$

   By $(\star\star)$ we know $\cos(y) > 0$ so that $\cos(y) = \sqrt{1-\sin^2(y)}$. So we have

   $$\frac{dy}{dx} = \frac{1}{\cos(y)} = \frac{1}{\sqrt{1-\sin^2(y)}} \overset{(\star)}{=} \frac{1}{\sqrt{1-x^2}}.$$

2. Prove that $\frac{d}{dx}\arccos(x) = \frac{-1}{\sqrt{1-x^2}}$.

   Proof: Set $y = \arccos(x)$ so that we want to show $\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$. By the definition of arccosine we know

   $$\boxed{x = \cos(y) \quad (\star)} \qquad \text{and} \qquad \boxed{0 \le y \le \pi \quad (\star\star)}$$

   Differentiating both sides of $(\star)$ with respect to x gives

   $$\frac{d}{dx}(x) = \frac{d}{dx}(\cos(y)) \quad \Longrightarrow \quad 1 = \frac{d}{dy}(\cos(y))\frac{dy}{dx}$$

   $$\Longrightarrow \quad 1 = -\sin(y)\frac{dy}{dx} \quad \Longrightarrow \quad \frac{dy}{dx} = \frac{-1}{\sin(y)}.$$

   Now let's use the pythagorean identity:

   $$\sin^2(y)+\cos^2(y) = 1 \quad \Longrightarrow \quad \sin^2(y) = 1-\cos^2(y) \quad \Longrightarrow \quad \sin(y) = \pm\sqrt{1-\cos^2(y)}$$

   By $(\star\star)$ we know $\sin(y) > 0$ so that $\sin(y) = \sqrt{1-\cos^2(y)}$. So we have

   $$\frac{dy}{dx} = \frac{-1}{\sin(y)} = \frac{-1}{\sqrt{1-\cos^2(y)}} \overset{(\star)}{=} \frac{-1}{\sqrt{1-x^2}}.$$

3. Prove that $\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$.

Proof: Set $y = \arctan(x)$ so that we want to show $\frac{dy}{dx} = \frac{1}{1+x^2}$. By the definition of arctangent we know

$$\boxed{x = \tan(y) \quad (\star)}$$

[Note: it is also true that $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, but we will not need this fact in the proof.] Differentiating both sides of $(\star)$ with respect to x gives

$$\frac{d}{dx}(x) = \frac{d}{dx}(\tan(y)) \quad \Longrightarrow \quad 1 = \frac{d}{dy}(\tan(y))\frac{dy}{dx}$$

$$\Longrightarrow \quad 1 = \sec^2(y)\frac{dy}{dx} \quad \Longrightarrow \quad \frac{dy}{dx} = \frac{1}{\sec^2(y)}.$$

Now let's use the pythagorean identity:

$$\sin^2(y)+\cos^2(y) = 1 \quad \Longrightarrow \quad \frac{\sin^2(y)}{\cos^2(y)}+\frac{\cos^2(y)}{\cos^2(y)} = \frac{1}{\cos^2(y)} \quad \Longrightarrow \quad \tan^2(y)+1 = \sec^2(y)$$

So we have

$$\frac{dy}{dx} = \frac{1}{\sec^2(y)} = \frac{1}{\tan^2(y)+1} \overset{(\star)}{=} \frac{1}{1+x^2}.$$

4. Prove that $\frac{d}{dx}\ln(x) = \frac{1}{x}$.

Proof: Set $y = \ln(x)$ so that we want to show $\frac{dy}{dx} = \frac{1}{x}$. By the definition of the natural logarithm we know

$$\boxed{x = e^y \quad (\star)}$$

Differentiating both sides of $(\star)$ with respect to x gives

$$\frac{d}{dx}(x) = \frac{d}{dx}(e^y) \quad \Longrightarrow \quad 1 = \frac{d}{dy}(e^y)\frac{dy}{dx} \quad \Longrightarrow \quad 1 = e^y\frac{dy}{dx}$$

$$\Longrightarrow \quad \frac{dy}{dx} = \frac{1}{e^y} \quad \overset{(\star)}{\Longrightarrow} \quad \frac{dy}{dx} = \frac{1}{x}.$$

5. Prove that $\frac{d}{dx}\log_b(x) = \frac{1}{x\ln(b)}$.

Proof: Set $y = \log_b(x)$ so that we want to show $\frac{dy}{dx} = \frac{1}{x\ln(b)}$. By the definition of a logarithm we know

$$\boxed{x = b^y \quad (\star)}$$

Differentiating both sides of $(\star)$ with respect to x gives

$$\frac{d}{dx}(x) = \frac{d}{dx}(b^y) \quad \implies \quad 1 = \frac{d}{dy}(b^y)\frac{dy}{dx} \quad \implies \quad 1 = b^y \ln(b)\frac{dy}{dx}$$

$$\implies \quad \frac{dy}{dx} = \frac{1}{b^y \ln(b)} \quad \overset{(\star)}{\implies} \quad \frac{dy}{dx} = \frac{1}{x \ln(b)}.$$

6. Let f be a differentiable function with inverse f^{-1} which is also differentiable. Prove that

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$$

as long as the denominator is not zero.

Proof: Set $y = f^{-1}(x)$ so that we want to show $\frac{dy}{dx} = \frac{1}{f'(f^{-1}(x))}$. By the definition of inverse functions we know

$$\boxed{x = f(y) \quad (\star)}$$

Differentiating both sides of $(\star)$ with respect to x gives

$$\frac{d}{dx}(x) = \frac{d}{dx}(f(y)) \quad \implies \quad 1 = \frac{d}{dy}(f(y))\frac{dy}{dx} \quad \implies \quad 1 = f'(y)\frac{dy}{dx}$$

$$\implies \quad \frac{dy}{dx} = \frac{1}{f'(y)} \quad \implies \quad \frac{dy}{dx} = \frac{1}{f'(f^{-1}(x))}.$$