Section 4 – 5: The Rational Zeros of a Polynomial

The **standard form** for a general **polynomial of degree n is written**

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + ...+ a_1 x + a_0$$

where the highest degree term is $a_n x^n$ and the lowest degree term is a_0

and $a_n x^n \neq 0$

Examples of a polynomial in standard form

$f(x) = 2x^4 + x^3 - 17x^2 + 6$ $\quad$ $f(x) = 4x^5 - x^4 + 2x - 7$ $\quad$ $f(x) = 3x^6 + x^4 - 4x^2 - 4$

Finding all the Possiable Rational Zeros of a Polynomial

The list of all possible zeros of f (x) $= \pm \dfrac{\text{all the factors of } a_n}{\text{all the factors of } a_0}$

Example 1

The list of all possiable factors of

$f(x) = x^4 + x^3 - 17x - 8$

$= \pm \dfrac{\text{all the factors of 8}}{\text{all the factors of 1}}$

$\pm\dfrac{1}{1}, \pm\dfrac{2}{1}, \pm\dfrac{4}{1}, \pm\dfrac{8}{1},$

reduce the fractions
and list in order
$\pm 1, \pm 2, \pm 4, \pm 8$

Example 2

The list of all possiable factors of

$f(x) = 5x^4 + x^3 - 17x^2 + 4$

$= \pm \dfrac{\text{all the factors of 4}}{\text{all the factors of 5}}$

$\pm\dfrac{1}{1}, \pm\dfrac{2}{1}, \pm\dfrac{4}{1}, \pm\dfrac{1}{5}, \pm\dfrac{2}{5}, \pm\dfrac{4}{5}$

reduce the fractions
and list in order
$\pm 1, \pm 2, \pm 4, \pm\dfrac{1}{5}, \pm\dfrac{2}{5}, \pm\dfrac{4}{5}$

Example 3

The list of all possiable factors of

$f(x) = 4x^5 + x^4 + x^3 - 17x - 4$

$= \pm \dfrac{\text{all the factors of 4}}{\text{all the factors of 4}}$

$\pm\frac{1}{1}, \pm\frac{1}{2}, \pm\frac{1}{4}, \pm\frac{2}{1}, \pm\frac{2}{2}, \pm\frac{2}{4}, \pm\frac{4}{1}, \pm\frac{4}{2}, \pm\frac{4}{4}$

reduce the fractions

$\pm 1, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm 2, \pm 1, \pm\frac{1}{2}, \pm 4, \pm 2, \pm 1$

eliminate the repeats and list the integers first

$\pm 1, \pm 2 \ \pm 4, \pm\frac{1}{2}$

Example 4

The list of all possiable factors of

$f(x) = 2x^4 + x^3 - 17x^2 + 6$

$= \pm \dfrac{\text{all the factors of 6}}{\text{all the factors of 2}}$

$\pm\frac{1}{1}, \pm\frac{1}{2}, \pm\frac{2}{1}, \pm\frac{2}{2}, \pm\frac{3}{1}, \pm\frac{3}{2}, \pm\frac{6}{1}, \pm\frac{6}{2}$

reduce the fractions

$\pm 1, \pm\frac{1}{2}, \pm 2, \pm 1, \pm 3, \pm\frac{3}{2}, \pm 6, \pm 3$

eliminate the repeats and list the integers first

$\pm 1, \pm 2 \ \pm 3, \pm 6, \pm\frac{1}{2}, \pm\frac{3}{2}$

eliminate

Finding all the Rational Roots for a Polynomial

Example 1

Find all the rational roots of

$f(x) = x^3 + 2x^2 - x - 2$

possiable roots

$= \pm \dfrac{\text{all the factors of 2}}{\text{all the factors of 1}}$

$\pm\frac{1}{1}\ ,\ \pm\frac{2}{1}\ ,\ \pm\frac{1}{2}\ ,\ \pm\frac{2}{2}$

reduce the fractions and list in order

$\pm 1\ ,\ \pm 2\ ,\ \pm\frac{1}{2}$

test 1 as a root

$$\begin{array}{r|rrrr} 1 & 1 & 2 & -1 & 2 \\ & \downarrow & 1 & 3 & 2 \\ \hline & 1 & 3 & 2 & (4) \end{array}$$

1 is not a root

test –1 as a root

$$\begin{array}{r|rrrr} -1 & 1 & 2 & -1 & 2 \\ & \downarrow & -1 & -1 & -2 \\ \hline & 1 & 1 & -2 & (0) \end{array}$$

0 is a root

and $x^2 + x - 2$ remains

$x^2 + x - 2$
factors into
$(x+2)(x-1)$
so $x^2 + x - 2$ has
roots of -2 and 1

The three rational roots of $f(x) = x^3 + 2x^2 - x - 2$ **are –1, 1, and 2**

and the factors of $f(x) = x^3 + 2x^2 - x - 2$ are (x+1) (x-1) (x-2)

Example 2

Find all the rational roots of

$f(x) = x^3 + 2x^2 - x - 2$

possiable roots

$= \pm \dfrac{\text{all the factors of 2}}{\text{all the factors of 1}}$

$\pm \frac{1}{1}, \pm \frac{2}{1}, \pm \frac{1}{2}, \pm \frac{2}{2}$

reduce the fractions and list in order

$\pm 1, \pm 2, \pm \frac{1}{2}$

Step 1

test 1 as a root

$$\begin{array}{r|rrrr} \underline{1} & 1 & 2 & -1 & 2 \\ & \downarrow & 1 & 3 & 2 \\ \hline & 1 & 3 & 2 & (4) \end{array}$$

1 is not a root

Step 2

test –1 as a root

$$\begin{array}{r|rrrr} \underline{-1} & 1 & 2 & -1 & 2 \\ & \downarrow & -1 & -1 & -2 \\ \hline & 1 & 1 & -2 & (0) \end{array}$$

0 is a root and

$x^2 + x - 2$ remains

Step 3

$x^2 + x - 2$

factors into

$(x + 2)(x - 1)$

so $x^2 + x - 2$ has

roots of -2 and 1

The three rational roots of $f(x) = x^3 + 2x^2 - x - 2$ **are –1, 1, and 2**

and the factors of $f(x) = x^3 + 2x^2 - x - 2$ are (x + 1) (x – 1) (x – 2)

Example 3

Find all the rational roots of

$f(x) = x^4 - 2x^3 - 7x^2 + 8x + 12$

possiable roots

$= \pm \dfrac{\text{all the factors of 12}}{\text{all the factors of 1}}$

$\pm\frac{1}{1}, \pm\frac{2}{1}, \pm\frac{3}{1}, \pm\frac{4}{1}, \pm\frac{6}{1}, \pm\frac{12}{1}$

reduce the fractions and list in order

±1 , ± 2 , ± 3, ± 4, ± 6

Step 1

test 1 as a root

1 |	1	−2	−7	8	12
	↓	1	−1	−8	0
	1	−1	−8	0	(12)

1 is not a root

Step 2

test –1 as a root

−1 |	1	−2	−7	8	12
	↓	−1	3	4	−12
	1	−3	−4	12	(0)

−1 is a root

and 1 −3 −4 12 remains

Step 2A

we now use

__ |	1	−3	−4	12
	↓			
	1			

to test for the remaining roots

Step 3

test 2 as a root

2 |	1	−3	−4	12
	↓	2	−2	−12
	1	−1	−6	(0)

2 is a root and

$x^2 - x - 6$ remains

Step 4

$x^2 - x - 6$

factors into

$(x+2)(x-3)$

so $x^2 - x - 6$ has

roots of 3 and −2

The four rational roots of $f(x) = x^4 - 2x^3 - 7x^2 + 8x + 12$ **are –1, 2, –2 and 3**

Example 3

Find all the rational roots of

$f(x) = 4x^4 + 4x^3 - 3x^2 - 2x + 1$

$$\text{possiable roots} = \pm \frac{\text{all the factors of 1}}{\text{all the factors of 4}}$$

$$\pm\frac{1}{1}, \pm\frac{1}{2}, \pm\frac{1}{4}$$

reduce the fractions and list in order

$$\pm 1, \pm\frac{1}{2}, \pm\frac{1}{4}$$

Step 1

test 1 as a root

1 |	4	12	13	6	1
	↓	4	16	29	35
	4	16	29	35	(36)

1 is not a root

Step 2

test –1 as a root

−1 |	4	12	13	6	1
	↓	−4	−8	−5	−1
	4	8	5	1	(0)

−1 is a root

and 4 8 5 1 remains

Step 2A

we now use

__ |	4	8	5	1
	↓			
	4			

to test for the remaining roots

Step 3

test $1/2$ as a root

1/2 |	4	8	5	1
	↓	2	5	5
	4	10	10	(6)

$1/2$ is not a root

Step 4

test $-1/2$ as a root

−1/2 |	4	8	5	1
	↓	−2	−3	−1
	4	6	2	(0)

$-1/2$ is a root and

$4x^2 + 6x + 2$ remains

Step 4

$4x^2 + 6x + 2$

factors into

$2(2x+1)(x+1)$

so $4x^2 + 6x + 2$ has

roots of $-1/2$ and -1

The four rational roots of $f(x) = 4x^4 + 12x^3 + 13x^2 + 6x + 1$ **are –1, –1, –1/2 and –1/2**

Note: If you had retested –1 in step 3 you you have found it was a root twice and had $4x^2 + 4x + 1$ left which factors into (2x + 1) (2x + 1) and not needed to test 1/2 or – 1/2