Section 12 – 3A: Geometric Sequences

Arithmetic Sequence

An arithmetic sequence is a discreet ordered list of terms that have a constant rate of growth. If we graph ordered pairs of \((n, a_n)\) for an arithmetic sequence, we would see that they are discrete points that lie on a line with the equation \(a_n = d \cdot n + c\). The line is formed by a continuous set of points but the arithmetic sequence are the discrete ordered pairs in red. Positive values for \(d\) produce a set of points that are contained in a line with a positive slope. Negative values for \(d\) produce a set of points that are contained in a line with a negative slope. The domain for the line is limited to positive integers so the points are only in the first quadrant.

The line \(a_n = d \cdot n + c\) models the arithmetic sequence but only the discreet set of points in red are part of the arithmetic sequence.

Arithmetic Sequences model discrete constant growth because the domain is limited to the positive integers (counting numbers).

The growth pattern could be called discrete linear growth but in this chapter we call this growth Arithmetic Growth if the terms in the sequence are increasing and Arithmetic Decay if the terms in the sequence are decreasing.

Geometric Sequences

A geometric sequence is a sequence that has a constant (labeled \(r\)) where the first term is MULTIPLIED by \(r\) to get the second term and then the second term is MULTIPLIED by \(r\) to get the third term. The process is continued for as long as you like.

This process creates a sequence where the ratio of any term and the one that precedes it is the constant \(r\). The sequence is labeled \(a_n\). The first term of the sequence is labeled \(a_1\) and the number multiplied by a term to get the next term is called the common ratio and is represented by the variable \(r\).
A Geometric Sequence can be defined recursively as

If \( a_1 \) is the first term and \( r \) is the common ratio then

\[
a_n = (a_{n-1}) \cdot r
\]

where \( r \) is the common ratio between terms

\[
r = \frac{a_n}{a_{n-1}} \quad \text{and} \quad r \neq -1, 0, 1
\]

**Example 1** \( a_1 > 0 \) and \( r > 1 \)

**Step 1:** Select any real number as the first term of the sequence.

Let \( a_1 = 3 \)

**Step 2:** Select any real number as the common ratio. Let \( r = 2 \)

**Step 3A:** Multiply the first term \( a_1 \) by 2 to get the second term

\[
a_2 = a_1 \cdot 2 = 3 \cdot 2 = 6
\]

**Step 3B:** Multiply the second term \( a_2 = 6 \) by 2 to get the third term

\[
a_3 = a_2 \cdot 2 = 6 \cdot 2 = 12
\]

**Step 3C:** Multiply the third term \( a_3 = 12 \) by 2 to get the 4th term

\[
a_4 = a_3 \cdot 2 = 12 \cdot 2 = 24
\]

**Step 3D:** Multiply the 4th term \( a_4 = 24 \) by 2 to get the 5th term

\[
a_5 = a_4 \cdot 2 = 24 \cdot 2 = 48
\]

**Step 3E:** Multiply the 5th term \( a_5 = 48 \) by 2 to get the 6th term

\[
a_6 = a_5 \cdot 2 = 48 \cdot 2 = 96
\]

continue multiplying the current term by the common ration to get the next term.

\[
\frac{3}{a_1}, \frac{6}{a_1}, \frac{12}{a_1}, \frac{24}{a_1}, \frac{48}{a_1}, \frac{96}{a_1}, \ldots
\]

3, 6, 12, 24, 48, 96, 192, ....

\( r \) was a positive number greater than 1 so the geometric sequence had terms that INCREASED in value
A Geometric Sequence can be defined recursively.

Example 2 \( a_1 > 0 \) and \( 0 < r < 1 \)

Step 1: Select any real number as the first term of the sequence.  
Let \( a_1 = 256 \)

Step 2: Select any real number as the common ratio. Let \( r = 1/2 \)

Step 3A: Multiply the first term \( a_1 \) by \( 1/2 \) to get the second term  
\[ a_2 = a_1 \cdot 1/2 = 256 \cdot 1/2 = 128 \]

Step 3B: Multiply the second term \( a_2 = 128 \) by \( 1/2 \) to get the third term  
\[ a_3 = a_2 \cdot 1/2 = 128 \cdot 1/2 = 64 \]

Step 3C: Multiply the third term \( a_3 = 64 \) by 2 to get the 4th term  
\[ a_4 = a_3 \cdot 1/2 = 64 \cdot 1/2 = 32 \]

Step 3D: Multiply the 4th term \( a_4 = 32 \) by \( 1/2 \) to get the 5th term  
\[ a_5 = a_4 \cdot 1/2 = 32 \cdot 1/2 = 16 \]

Step 3E: Multiply the 5th term \( a_5 = 16 \) by \( 1/2 \) to get the 6th term  
\[ a_6 = a_5 \cdot 1/2 = 16 \cdot 1/2 = 8 \]

continue multiplying the current term by the common ratio to get the next term.

\[
\begin{align*}
256 & \quad , & 128 & \quad , & 64 & \quad , & 32 & \quad , & 16 & \quad , & 8 \\
& 256 \cdot \frac{1}{2} & , & 128 \cdot \frac{1}{2} & , & 64 \cdot \frac{1}{2} & , & 32 \cdot \frac{1}{2} & , & 16 \cdot \frac{1}{2} & , & \ldots
\end{align*}
\]

\[
\begin{align*}
256 & \quad , & 128 & \quad , & 64 & \quad , & 32 & \quad , & 16 & \quad , & 8 & \quad , & 4 & \quad , & 2 & \quad , & 1 & \quad , & \frac{1}{2} & \quad , & \frac{1}{4} & \quad , & \frac{1}{8} & \quad , & \ldots
\end{align*}
\]

\( a_1 \) was a positive number \( 0 < r < 1 \) so the geometric sequence had terms that DECREASED in value.
When \( a_1 > 0 \) and \( r > 1 \) the sequence is modeled by an increasing exponential function with positive integers for values for the domain. We say that the terms display Geometric Growth.

\[
a_n = a_1 \cdot r^{n-1} \quad a_1 > 0 \quad r > 1
\]

Arithmetic Growth

\[
a_n = dx + c \quad d > 0
\]

When \( a_1 > 0 \) and \( 0 < r < 1 \) the sequence is modeled by a decreasing exponential function with positive integers for values for the domain. We say that the terms display Geometric Decay.

\[
a_n = a_1 \cdot r^{n-1} \quad a_1 > 0 \quad 0 < r < 1
\]

Arithmetic Decay

\[
a_n = d \cdot n + c \quad d < 0
\]
Why is there a restriction that $r \neq -1, 0, 1$

The requirement that $r \neq -1, 0, 1$ is based on a requirement that geometric sequences display a discrete version of a continuous exponential growth or decay pattern.

When we create a sequence with $r = -1, 0, 1$ the resulting sequence does NOT display an exponential growth or decay pattern. For this reason we limit the values of $r$ by stating $r \neq -1, 0, 1$

<table>
<thead>
<tr>
<th>First Term</th>
<th>$R$</th>
<th>Type of Sequence</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$ is positive</td>
<td>$R = 1$</td>
<td>Constant Not geometric</td>
<td>$a_n = 2, 2, 2, 2, 2, ...$</td>
</tr>
<tr>
<td>$a_1$ is positive</td>
<td>$R = 0$</td>
<td>no pattern Not geometric</td>
<td>$a_n = 2, 0, 0, 0, 0, ..., $</td>
</tr>
<tr>
<td>$a_1$ is positive</td>
<td>$R = -1$</td>
<td>Alternating Not geometric</td>
<td>$a_n = 2, -2, 2, -2, 2, -2, ...$</td>
</tr>
<tr>
<td>$a_1$ is negative</td>
<td>$R = 1$</td>
<td>Constant Not geometric</td>
<td>$a_n = -2, -2, -2, -2, -2, ...$</td>
</tr>
<tr>
<td>$a_1$ is negative</td>
<td>$R = 0$</td>
<td>no pattern Not geometric</td>
<td>$a_n = -2, 0, 0, 0, 0, ..., $</td>
</tr>
<tr>
<td>$a_1$ is negative</td>
<td>$R = -1$</td>
<td>Alternating Not geometric</td>
<td>$a_n = -2, 2, -2, 2, -2, ...$</td>
</tr>
</tbody>
</table>

The value the first term $a_1$ and the common ratio $r$ determine the nature of the geometric sequence.

**If $r > 1$ and $a_1$ is positive** then the discrete geometric sequence models exponential growth.

The table below shows several values for pairs for $a_n = 2^n$

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_n$</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>16</td>
<td>32</td>
<td>64</td>
<td>128</td>
<td>256</td>
</tr>
</tbody>
</table>

**If $0 < r < 1$ and $a_1$ is positive** then the discrete geometric sequence models exponential decay.

The table below shows several values for pairs for $a_n = (1/2)^n$

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_n$</td>
<td>1/2</td>
<td>1/4</td>
<td>1/8</td>
<td>1/16</td>
<td>1/32</td>
<td>1/64</td>
<td>1/128</td>
<td>1/256</td>
</tr>
</tbody>
</table>
Geometric sequences with a common ratio $r$ that is negative $a_1 > 0$ and $r < 0$

The common ratio $r$ of a sequence may be negative. That sequence will have terms that alternate between positive and negative values.

$$a_n = 2, -4, 8, -16, 32, -64, 128, -256, \ldots \text{ where } a_1 > 0 \text{ and } r < 0$$

We call these sequences altering geometric sequences.

The sequence has two parts. The positive numbers $a_n = 2, 8, 32, 128, \ldots$ display geometric growth and the negative values $a_n = -4, -16, -64, -256, \ldots$ display geometric decay.

### Possible Geometric sequences if the first term is $a_1$ positive

<table>
<thead>
<tr>
<th>First Term</th>
<th>R</th>
<th>Type of Sequence</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$ is positive</td>
<td>$R &gt; 1$</td>
<td>Increasing Geometric</td>
<td>$a_1 = 1$ and $r = 2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$a_n = 1, 2, 4, 8, 16, 32, \ldots$</td>
</tr>
<tr>
<td>$a_1$ is positive</td>
<td>$0 &lt; R &lt; 1$</td>
<td>Decreasing Geometric</td>
<td>$a_1 = 4$ and $r = 1/2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$a_n = 4, 2, 1, \frac{1}{2}, \frac{1}{4}, \ldots$</td>
</tr>
<tr>
<td>$a_1$ is positive</td>
<td>$-1 &lt; R &lt; 0$</td>
<td>Alternating Geometric</td>
<td>$a_1 = 3$ and $r = 1/3$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$a_n = 3, -1, \frac{1}{3}, -\frac{1}{9}, \frac{1}{27}, -\frac{1}{81}, \ldots$</td>
</tr>
<tr>
<td>$a_1$ is positive</td>
<td>$-1 &lt; R$</td>
<td>Alternating Geometric</td>
<td>$a_1 = 3$ and $r = -2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$a_n = 3, -6, 12, -24, 48, -96, \ldots$</td>
</tr>
</tbody>
</table>

### Possible sequences if the first term is $a_1$ negative

<table>
<thead>
<tr>
<th>First Term</th>
<th>R</th>
<th>Type of Sequence</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$ is negative</td>
<td>$R &gt; 1$</td>
<td>Alternating Geometric</td>
<td>$a_1 = -2$ and $r = 3$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$a_n = -2, 6, -18, 54, -162, \ldots$</td>
</tr>
<tr>
<td>$a_1$ is negative</td>
<td>$0 &lt; R &lt; 1$</td>
<td>Decreasing Geometric</td>
<td>$a_1 = -4$ and $r = 1/2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$a_n = -4, -2, -1, -\frac{1}{2}, -\frac{1}{4}, \ldots$</td>
</tr>
<tr>
<td>$a_1$ is negative</td>
<td>$-1 &lt; R &lt; 0$</td>
<td>Alternating Geometric</td>
<td>$a_1 = -4$ and $r = -1/2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$a_n = -4, 2, -1, \frac{1}{2}, -\frac{1}{4}, \ldots$</td>
</tr>
<tr>
<td>$a_1$ is negative</td>
<td>$-1 &lt; R$</td>
<td>Alternating Geometric</td>
<td>$a_1 = -3$ and $r = -2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$a_n = -3, 6, -12, 24, -48, 96, \ldots$</td>
</tr>
</tbody>
</table>
An Geometric Sequence can be defined recursively as

\( a_1 \) is the first term and \( r \) is the common ratio where \( a_1 \) and \( r \) are real numbers.

\[ a_1 = \text{the first term} \quad \text{and} \quad a_n = r \cdot a_{n-1} \text{ for } n > 1 \]

**Step 1:** Select any real number as the first term of the sequence.

Let \( a_1 = 2 \)

**Step 2:** Select any real number as the common ratio. Let \( r = 3 \)

**Step 3A:** Multiply the first term \( a_1 = 2 \) by 3 to get the second term

\[ a_2 = 3 \cdot a_1 = 3 \cdot 2 = 6 \]

**Step 3B:** Multiply the second term \( a_2 = 6 \) by 3 to get the third term

\[ a_3 = 3 \cdot a_2 = 3 \cdot 6 = 18 \]

**Step 3C:** Multiply the third term \( a_3 = 18 \) by 3 to get the 4th term

\[ a_4 = 3 \cdot a_3 = 3 \cdot 18 = 54 \]

**Step 3D:** Multiply the 4th term \( a_4 = 54 \) by 3 to get the 5th term

\[ a_5 = 3 \cdot a_4 = 3 \cdot 54 = 162 \]

**Step 3E:** Multiply the 5th term \( a_5 = 162 \) by 3 to get the 6th term

\[ a_6 = 3 \cdot a_5 = 3 \cdot 162 = 486 \]

continue adding the common difference \( d \) to the current term to get the next term.

\[
\begin{align*}
\frac{2}{a_1} & , \frac{6}{3 \cdot 2} , \frac{18}{3 \cdot 6} , \frac{54}{3 \cdot 18} , \frac{162}{3 \cdot 54} , \frac{486}{3 \cdot 162} \\
& \quad a_n = 2 \cdot 6 \cdot 18 \cdot 54 \cdot 162 \cdot 486 \quad ...... 
\end{align*}
\]
Find the first 5 terms of the geometric sequence.

**Example 1**

\(a_1 = 1\) and \(r = 3\)
\(a_n = a_{n-1} \cdot 3\)

\(a_1 = 1\)
\(a_2 = a_1 \cdot 3 = 1 \cdot 3 = 3\)
\(a_3 = a_2 \cdot 3 = 3 \cdot 3 = 9\)
\(a_4 = a_3 \cdot 3 = 9 \cdot 3 = 27\)
\(a_5 = a_4 \cdot 3 = 27 \cdot 3 = 81\)

1 , 3 , 9 , 27 , 81 , ....

**Example 2**

\(a_1 = 4\) and \(r = 2\)
\(a_n = a_{n-1} \cdot 2\)

\(a_1 = 4\)
\(a_2 = a_1 \cdot 2 = 4 \cdot 2 = 8\)
\(a_3 = a_2 \cdot 2 = 8 \cdot 2 = 16\)
\(a_4 = a_3 \cdot 2 = 16 \cdot 2 = 32\)
\(a_5 = a_4 \cdot 2 = 32 \cdot 2 = 64\)

4 , 8 , 16 , 32 , 64 , ....

**Example 3**

The common ratio can be **negative**.

\(a_1 = 5\) and \(r = -2\)
\(a_n = a_{n-1} \cdot (-2)\)

\(a_1 = 5\)
\(a_2 = a_1 \cdot (-2) = 5 \cdot (-2) = -10\)
\(a_3 = a_2 \cdot (-2) = -10 \cdot (-2) = 20\)
\(a_4 = a_3 \cdot (-2) = 20 \cdot (-2) = -40\)
\(a_5 = a_4 \cdot (-2) = -40 \cdot (-2) = 80\)

5 , -10 , 20 , -40 , 80 , ....

**Example 4**

The common ratio can be a **fraction**.

\(a_1 = 16\) and \(r = \frac{1}{2}\)
\(a_n = a_{n-1} \cdot \frac{1}{2}\)

\(a_1 = 16\)
\(a_2 = a_1 \cdot \frac{1}{2} = 16 \cdot \frac{1}{2} = 8\)
\(a_3 = a_2 \cdot \frac{1}{2} = 8 \cdot \frac{1}{2} = 4\)
\(a_4 = a_3 \cdot \frac{1}{2} = 4 \cdot \frac{1}{2} = 2\)
\(a_5 = a_4 \cdot \frac{1}{2} = 2 \cdot \frac{1}{2} = 1\)

16 , 8 , 4 , 2 , 1 , .... ,

**Note:** It is common to **NOT REDUCE** the fractions so the pattern of the sequence is easier to see.
Determining if a sequence is geometric.

The common ratio \( r \) is found by \( r = \frac{a_n}{a_{n-1}} \)

\( r \) must be the same for ALL PAIRS of consecutive terms in the sequence and \( r \neq -1, 0, 1 \).

Assume the pattern shown continues. Is the sequence shown geometric?

**Example 1**

1, 4, 16, 64, 256, ....

\[
\begin{align*}
  r &= \frac{256}{64} = 4 \\
  r &= \frac{64}{16} = 4 \\
  r &= \frac{16}{4} = 4 \\
  r &= \frac{4}{1} = 4 \\
  r &= 4
\end{align*}
\]

The sequence is Geometric

**Example 2**

81, 27, 9, 3, 1, ....

\[
\begin{align*}
  r &= \frac{1}{3} = \frac{1}{3} \\
  r &= \frac{3}{9} = \frac{1}{3} \\
  r &= \frac{9}{27} = \frac{1}{3} \\
  r &= \frac{27}{81} = \frac{1}{3} \\
  r &= \frac{1}{3}
\end{align*}
\]

The sequence is Geometric

**Example 3**

5, 5, 5, 5, 5, ....

\[
\begin{align*}
  r &= \frac{5}{5} = 1 \\
  r &= \frac{5}{5} = 1 \\
  r &= \frac{5}{5} = 1 \\
  r &= \frac{5}{5} = 1 \\
  r &= \frac{5}{5} = 1 \\
  r &= 1
\end{align*}
\]

r \neq 1

The sequence is NOT Geometric

**Example 4**

\[
\begin{align*}
  \frac{2}{1}, \frac{4}{3}, \frac{8}{9}, \frac{16}{27}, \ldots
\end{align*}
\]

\[
\begin{align*}
  r &= \frac{16}{27} = \frac{16}{27} \cdot \frac{9}{8} = \frac{2}{3} \\
  r &= \frac{8}{9} = \frac{8}{9} \cdot \frac{3}{4} = \frac{2}{3} \\
  r &= \frac{9}{16} = \frac{9}{16} \cdot \frac{1}{8} = \frac{2}{3} \\
  r &= \frac{4}{3} = \frac{4}{3} \cdot \frac{1}{2} = \frac{2}{3} \\
  r &= \frac{2}{3}
\end{align*}
\]

The sequence is Geometric
A formula for the nth term of an arithmetic sequence

An geometric sequence has a first term $a_1$ and each term thereafter is MULTIPLIED by the same constant. The constant that each term is multiplied by is called the **common ratio**.

An **Geometric Sequence** can be defined recursively as

If $a_1 = \text{the first term}$ and $r$ is the common ratio and $r \neq -1, 0, 1$ then

$$a_n = (a_{n-1}) \cdot r$$

The recursive form requires that you find any term by adding the common difference $d$ to the term just before it. To find the 33rd term you would need the 32nd term and that requires the 31st term and so on. It is always better to have a formula that is in closed form. That allows you to find the 33rd term by plugging $n = 33$ directly into the formula. This finds the 33rd term in one step.

**Finding a closed form formula for the nth term of an arithmetic sequence.**

Start with a first term and a common ratio $r$

$$a_1, \ (a_1) \cdot r, \ (a_1 \cdot r^1) \cdot r, \ (a_1 \cdot r^2) \cdot r, \ (a_1 \cdot r^3) \cdot r, \ .....$$

$$a_1, \ a_1 \cdot r^1, \ a_1 \cdot r^2, \ a_1 \cdot r^3, \ a_1 \cdot r^4, \ ..... , \ a_1 \cdot r^{n-1}$$

$n = 1 \quad n = 2 \quad n = 3 \quad n = 4 \quad n = 5$

1st \quad 2nd \quad 3rd \quad 4th \quad 5th \quad n th
term \quad term \quad term \quad term \quad term$

The 2nd term is $a_1 \cdot r^1$, the 3rd term is $a_1 \cdot r^2$, the 4th term is $a_1 \cdot r^3$

You can see that each term is found by multiplying $a_1$ by one less $r$ then the number of the term. The $n$th term found by multiplying $a_1$ by one less $r$ then the number of the term. The $n$th term is found by multiplying $a_1$ by $r^{n-1}$

A Geometric Sequence can be written as

$$a_n = a_1 \cdot r^{n-1}$$

$$a_1, \ a_1r, \ a_1r^2, \ a_1r^3, \ .......... , a_1r^{n-1}$$
A closed form formula for the nth term of a geometric sequence.

For a Geometric Sequence whose first term is \( a_1 \) and whose common ratio between terms is \( r \), the n\(^{\text{th}} \) term is found by the formula

\[
a_n = a_1 \cdot r^{n-1}
\]

**Finding the first 5 terms of a Geometric Sequence**

**Example 1**

\( a_1 = 3 \) and \( r = 4 \)

\[
a_n = a_1 \cdot r^{n-1}
\]

\[
\begin{align*}
a_1 &= 3 \\
a_2 &= 3 \cdot 4^{2-1} = 3 \cdot 4^1 = 12 \\
a_3 &= 3 \cdot 4^{3-1} = 3 \cdot 4^2 = 48 \\
a_4 &= 3 \cdot 4^{4-1} = 3 \cdot 4^3 = 192 \\
a_5 &= 3 \cdot 4^{5-1} = 3 \cdot 4^4 = 768 \\
\end{align*}
\]

\( 3, 12, 48, 198, 768, \ldots \)

**Example 3**

\( a_1 = 81 \) and \( r = \frac{-1}{3} \)

\[
a_n = a_1 \cdot r^{n-1}
\]

\[
\begin{align*}
a_1 &= 81 \\
a_2 &= 81 \cdot \left(\frac{-1}{3}\right)^{2-1} = 81 \cdot \left(\frac{-1}{3}\right)^1 = -27 \\
a_3 &= 81 \cdot \left(\frac{-1}{3}\right)^{3-1} = 81 \cdot \left(\frac{-1}{3}\right)^2 = 9 \\
a_4 &= 81 \cdot \left(\frac{-1}{3}\right)^{4-1} = 81 \cdot \left(\frac{-1}{3}\right)^3 = -3 \\
a_5 &= 81 \cdot \left(\frac{-1}{3}\right)^{5-1} = 81 \cdot \left(\frac{-1}{3}\right)^4 = 1 \\
\end{align*}
\]

\( 81, -27, 9, -3, 1, \ldots \)

**Example 2**

\( a_1 = 4 \) and \( r = -2 \)

\[
a_n = a_1 \cdot r^{n-1}
\]

\[
\begin{align*}
a_1 &= 4 \\
a_2 &= 4 \cdot (-2)^{2-1} = 4 \cdot (-2)^1 = -8 \\
a_3 &= 4 \cdot (-2)^{3-1} = 4 \cdot (-2)^2 = 16 \\
a_4 &= 4 \cdot (-2)^{4-1} = 4 \cdot (-2)^3 = -32 \\
a_5 &= 4 \cdot (-2)^{5-1} = 4 \cdot (-2)^4 = 64 \\
\end{align*}
\]

\( 4, -8, 16, -32, 64, \ldots \)

**Example 4**

\( a_1 = 24 \) and \( r = \frac{2}{3} \)

\[
a_n = a_1 \cdot r^{n-1}
\]

\[
\begin{align*}
a_1 &= 24 \\
a_2 &= 24 \cdot \left(\frac{2}{3}\right)^{2-1} = 24 \cdot \left(\frac{2}{3}\right)^1 = 16 \\
a_3 &= 24 \cdot \left(\frac{2}{3}\right)^{3-1} = 24 \cdot \left(\frac{2}{3}\right)^2 = \frac{32}{3} \\
a_4 &= 24 \cdot \left(\frac{2}{3}\right)^{4-1} = 24 \cdot \left(\frac{2}{3}\right)^3 = \frac{64}{9} \\
a_5 &= 24 \cdot \left(\frac{2}{3}\right)^{5-1} = 24 \cdot \left(\frac{2}{3}\right)^4 = \frac{128}{27} \\
\end{align*}
\]

\( 24, 16, \frac{32}{3}, \frac{64}{9}, \frac{128}{27}, \ldots \)
Finding a specific term of a Geometric Sequence.

**Example 1**

If \( a_1 = 4 \) and \( r = 3 \)

\[ a_n = a_1 \cdot r^{n-1} \]

Find the 12th term

\[ a_{12} = 4 \cdot 3^{11} \]

**Example 2**

If \( a_1 = 2 \) and \( r = \frac{3}{4} \)

\[ a_n = a_1 \cdot r^{n-1} \]

Find the 10th term

\[ a_{10} = 2 \cdot \left( \frac{3}{4} \right)^9 \]

**Note:** Most calculators cannot display a number to a power if the power is greater that about 25. In those the calculator truncates the answer and replaces the digits with zeros and gives a answer in scientific notation form. Except for very small powers of a small base like \( 2^4 \) leaving the answer in exponential form is OK

**Finding what term of an Geometric Sequence has a given value**

If \( a_n = 2(3)^{n-1} \)

What term has a value of 486

If \( a_n = 2(3)^{n-1} = 486 \)

\[ 2(3)^{n-1} = 486 \]

solve for \( n \)

\[ (3)^{n-1} = 243 \]

\[ (3)^{n-1} = 3^5 \]

\( n - 1 = 5 \)

\( n = 6 \)
Exponential Growth Models

Exponential growth models of physical phenomena only apply within limited regions, as unbounded growth is not physically realistic. Although growth may initially be exponential, the modeled phenomena will eventually enter a region in which previously ignored negative feedback factors become significant or other underlying assumptions of the exponential growth model, such as continuity or instantaneous feedback, break down. In many of these cases the new pattern changes to a logarithmic growth model.

**Bacteria** exhibit exponential growth under optimal conditions. The number of microorganisms in a culture will increase exponentially until an essential nutrient is exhausted. Typically the first organism splits into two daughter organisms, who then each split to form four, who split to form eight, and so on. A virus (for example SARS, or smallpox) typically will spread exponentially at first, if no artificial immunization is available. Each infected person can infect multiple new people.

**Human population** exhibit exponential growth under optimal conditions. For example, according to the United States Census Bureau, over the last 100 years (1910 to 2010), the population of the United States of America is exponentially increasing at an average rate of one and a half percent a year (1.5%). This means that the doubling time of the American population (depending on the yearly growth in population) is approximately 50 years.

**Economic growth** is expressed in percentage terms, implying exponential growth. For example, U.S. GDP per capita has grown at an exponential rate of approximately two percent since 1945.

**Compound interest** at a constant interest rate provides exponential growth of the capital.

**Pyramid schemes** or Ponzi schemes also show this type of growth resulting in high profits for a few initial investors and losses among great numbers of investors.

**Processing power of computers.** Over the history of computing hardware, the number of transistors on integrated circuits doubles approximately every two years. The law is named after Intel cofounder Gordon E. More, who described the trend in his 1965 paper.

**Breakdown within a dielectric material.** A free electron becomes sufficiently accelerated by an externally applied electrical field that it frees up additional electrons as it collides with atoms or molecules of the dielectric media. These secondary electrons also are accelerated, creating larger numbers of free electrons. The resulting exponential growth of electrons and ions may rapidly lead to complete dielectric breakdown of the material.

**Nuclear chain reaction.** Each uranium nucleus that undergoes fission produces multiple neutrons, each of which can be absorbed by adjacent uranium atoms, causing them to fission in turn. The production rate of neutrons and induced uranium fission's increases exponentially, in an uncontrolled reaction. "Due to the exponential rate of increase, at any point in the chain reaction 99% of the energy will have been released in the last 4.6 generations. It is a reasonable approximation to think of the first 53 generations as a latency period leading up to the actual explosion, which only takes 3–4 generations."
**Electroacoustic amplification.** Positive feedback within the linear range of electrical or electroacoustic amplification can result in the exponential growth of the amplified signal, although resonance effects may favor some component frequencies of the signal over others.