Section 7 – 6: Complex Numbers

There is no Real Number that represents the square root of a negative number. No real number times itself can equal –9 so $\sqrt{-9}$ does not exist as a **Real Number**. This fact means that equations like $\sqrt{x} = -9$ do not have a real number solution. Past section have listed the answer to $\sqrt{x} = -9$ as NO Real Number or NRN.

This section will develop a number system that **does allow the the square root of a negative number.** This number system is called the **Complex Numbers**. The basic unit of this system is the imaginary unit.

The Imaginary Unit

The Imaginary Unit is denoted by the lower case letter i.
the imaginary number i is defined by the property that its square root is –1.

$i = \sqrt{-1}$ which can also be stated as $i^2 = -1$

Reducing an Imaginary Number written as $\sqrt{-a}$

If a is a positive number then $\sqrt{-a}$ can be reduced by writing $-a$ as the product of $\sqrt{-1}$ times $\sqrt{a}$ and replacing the $\sqrt{-1}$ with the imaginary number i

If a is a positive number then

$$\sqrt{-a}$$
$$= \sqrt{-1} \bullet \sqrt{a}$$
$$= i \bullet \sqrt{a}$$

Example 1

Simplify $\sqrt{-9}$

$= \sqrt{-1} \bullet \sqrt{9}$
$= i \bullet 3$
$= 3i$

Example 2

Simplify $3\sqrt{-36}$

$= 3 \bullet \sqrt{-1} \bullet \sqrt{36}$
$= 3 \bullet i \bullet 6$
$= 18i$

Example 3

Simplify $-\sqrt{-49}$

$= -1 \bullet \sqrt{-1} \bullet \sqrt{49}$
$= -1 \bullet i \bullet 7$
$= -7i$

Example 4

Simplify $\sqrt{-8}$

$= \sqrt{-1} \bullet \sqrt{8}$
$= i \bullet 2\sqrt{2}$
$= 2\sqrt{2}\ i$

Example 5

Simplify $\sqrt{-20}$

$= \sqrt{-1} \bullet \sqrt{20}$
$= i \bullet 2\sqrt{5}$
$= 2\sqrt{5}\ i$

Multiplying Imaginary Numbers written in terms of i

The product of the imaginary number i times itself is written as $i \bullet i$ or i^2 and $i^2 = -1$

If a and b are real numbers then

$$ai \bullet bi$$
$$= a \bullet b \bullet i^2$$
$$= a \bullet b \bullet (-1)$$
$$= -a \bullet b$$

Example 6

$(2i)(3i)$

$= 6 \bullet i^2$

$= 6(-1)$

$= -6$

Example 7

$(-5i)(4i)$

$= -20 \bullet i^2$

$= -20(-1)$

$= 20$

Example 8

$(-2i)(-5i)$

$= 10 \bullet i^2$

$= 10(-1)$

$= -10$

Multiplying Imaginary Numbers written in terms of $\sqrt{-a} \bullet \sqrt{-b}$

The product rule for radicals that we used in past sections required that the numbers under the radical sign be positive numbers. If a and b are positive numbers then $\sqrt{a} \bullet \sqrt{b} = \sqrt{a \bullet b}$
This product rule does not work when multiplying Imaginary Numbers. We need to reduce each imaginary number to an expression with i times the square root of a positive number. After that we can use the product rule $\sqrt{a} \bullet \sqrt{b} = \sqrt{a \bullet b}$ and the definition for $i^2 = -1$ to simplify the product.

Example 9

Simply

$\sqrt{-4} \bullet \sqrt{-16}$

$= i\sqrt{4} \bullet i\sqrt{16}$

$= i^2\sqrt{64}$

$= -8$

Example 10

Simply

$\sqrt{-2} \bullet \sqrt{-7}$

$= i\sqrt{2} \bullet i\sqrt{7}$

$= i^2\sqrt{14}$

$= -\sqrt{14}$

Example 11

Simply

$\sqrt{-4} \bullet \sqrt{-5}$

$= i\sqrt{4} \bullet i\sqrt{5}$

$= i^2 \bullet 2 \bullet \sqrt{5}$

$= -2\sqrt{5}$

Example 12

Simply

$\sqrt{-4} \bullet \sqrt{-5}$

$= i\sqrt{4} \bullet i\sqrt{5}$

$= i^2 \bullet 2 \bullet \sqrt{5}$

$= -2\sqrt{5}$

Example 13

Simply

$\sqrt{-12} \bullet \sqrt{-3}$

$= i\sqrt{12} \bullet i\sqrt{3}$

$= i^2\sqrt{36}$

$= -6$

Complex Numbers

A **Complex Number** is a number that can be written in the form $a+bi$ where a and b are real numbers. The a term is the real number part of the complex number and the bi term is the imaginary part of the complex number.

Adding and Subtracting Complex Numbers

If $a+bi$ and $c+di$ are complex numbers then their **sum** is written as $(a+bi)+(c+di)$ **+** $c+di$

$$(a+bi)+(c+di)$$
$$=a+bi+c+di$$
$$=a+c+(b+d)i$$

If $a+bi$ and $c+di$ are complex numbers then their **difference** is $a+bi-(c+di)$

$$(a+bi)-(c+di)$$
$$=a+bi-c-di$$
$$=(a-c)+(b-d)i$$

Example 14

Simply into $a+bi$ form

$(4+2i)+(7-6i)$

$=4+7+2i-6i$

$=11-4i$

Example 15

Simply into $a+bi$ form

$(5-2i)-(8-7i)$

$=5-2i-8+7i$

$=-3+5i$

Example 16

Simply into $a+bi$ form

$(-4+3i)-(2-5i)$

$=-4+3i-2+5i$

$=-6+8i$

Distributing Complex Numbers

Example 17

Simply into $a+bi$ form

$2i(5-3i)$

$=10i-6i^2$

$=10i-6(-1)$

$=10i+6$

$=6+10i$

Example 18

Simply into $a+bi$ form

$-4i(3+2i)$

$=-12i-8i^2$

$=-12i-8(-1)$

$=-12i+9$

$=9-12i$

Example 19

Simply into $a+bi$ form

$-i(-3+5i)$

$=3i-5i^2$

$=3i-5(-1)$

$=3i+5$

$=5+3i$

FOILing Complex Numbers

Example 20

Simply into $a+bi$ form

$(3+i)\ (3+2i)$

$= 9+6i+3i+2i^2$

$= 9+6i+3i+2(-1)$

$= 9+6i+3i-2$

$= 11+9i$

Example 21

Simply into $a+bi$ form

$(4+3i)\ (4-3i)$

$= 16-12i+12i-9i^2$

$= 16-12i+12i-9(-1)$

$= 16-12i+12i+9$

$= 25$

Example 22

Simply into $a+bi$ form

$(2-5i)\ (2+5i)$

$= 4-10i+10i-25i^2$

$= 4-10i+10i-5(-1)$

$= 4-10i+10i+5$

$= 9$

Reducing Fractions with Complex Numbers in the denominator

Example 23

Simply $\frac{2}{3i}$

i or $\sqrt{-1}$ is not allowed in the denominator

multiply the numerator and denominator by i

$\frac{2}{3i}\bullet\frac{i}{i}$

$= \frac{2i}{3i^2}$

$= \frac{2i}{3(-1)}$

$= \frac{2i}{-3}$

$= \frac{-2i}{3}$

Example 24

Simply $\frac{2}{4+3i}$

i or $\sqrt{-1}$ is not allowed in the denominator

multiply the numerator and denominator by $4-3i$

$\frac{2}{4+3i}\bullet\frac{4-3i}{4-3i}$

$= \frac{2(4-3i)}{16-12i+12i-9i^2}$

$= \frac{2(4-3i)}{16-9(-1)}$

$= \frac{2(4-3i)}{16+9}$

$= \frac{2(4-3i)}{25}$

Example 25

Simply $\frac{6}{3-2i}$

i or $\sqrt{-1}$ is not allowed in the denominator

multiply the numerator and denominator by $3+2i$

$\frac{6}{3-2i}\bullet\frac{3+2i}{3+2i}$

$= \frac{6(3+2i)}{9+6i-6i-4i^2}$

$= \frac{2(3+2i)}{16-4(-1)}$

$= \frac{2(3+2i)}{16+4}$

$= \frac{2(3+2i)}{20} = \frac{(3+2i)}{10}$

Optional Material

i raised to an even power $i^{\text{even power}}$

Example 1

Simply i^8

$= \left(i^2\right)^4$

$= (-1)^4$

$= 1$

Example 2

Simply i^{12}

$= \left(i^2\right)^6$

$= (-1)^6$

$= 1$

Example 3

Simply i^{14}

$= \left(i^2\right)^7$

$= (-1)^7$

$= -1$

i raised to an odd power $i^{\text{odd power}}$

Example 4

Simply i^5

$= i^{4+1}$

$= i^4 \bullet i^1$

$= \left(i^2\right)^2 \bullet i^1$

$= (-1)^2 \bullet i$

$= +i$

Example 5

Simply i^{11}

$= i^{10+1}$

$= i^{10} \bullet i^1$

$= \left(i^2\right)^5 \bullet i$

$= (-1)^5 \bullet i$

$= -i$

Example 6

Simply i^7

$= i^{6+1}$

$= i^6 \bullet i^1$

$= \left(i^2\right)^3 \bullet i$

$= (-1)^3 \bullet i$

$= -i$

