Rationalizing the Denominator of a Fraction with a Binomial Expression where one or both terms contains a Square Root

Each of following factions has a binomial expression in the denominator. Each denominator has a square root in one or both of its terms .

Example 1

$$\frac{3}{4-\sqrt{3}}$$

Example 2

$$\frac{5}{\sqrt{7}-2}$$

Example 3

$$\frac{3}{\sqrt{7}-\sqrt{3}}$$

It is common to require that the denominator not contain any radicals. The process of eliminating radicals from the denominator of a fraction is called **rationalizing the denominator**. In the previous section we had a monomial term with a square root in the denominator. In that case we multiplied the top and bottom of the fraction by the square root. This technique will not work with a binomial.

Multiplying the fraction by $\frac{\sqrt{3}}{\sqrt{3}}$ does not rationalize the denominator

$$\frac{5}{4-\sqrt{3}} \bullet \frac{\sqrt{3}}{\sqrt{3}} = \frac{5}{4\sqrt{3}-\sqrt{9}} = \frac{5}{4\sqrt{3}-3}$$

multiplying the bottom by $\sqrt{3}$ simply changes which term in the denominator has the square root.

The Square Root Conjugate

When we FOIL some binomials we end up with a whole number. This happens if the binomials are the **sum and difference of the same terms**. The pair is called a **Square Root Conjugate**

Example 4

$$\left(5+\sqrt{3}\right)\left(5-\sqrt{3}\right)$$

$= 25-2\sqrt{3}+2\sqrt{3}-\sqrt{9}$

the middle terms cancel

$= 25-\sqrt{9} = 25-3$

$= 22$

Example 5

$$\left(6-\sqrt{5}\right)\left(6+\sqrt{5}\right)$$

$= 36-7\sqrt{5}+7\sqrt{5}-\sqrt{25}$

the middle terms cancel

$= 36-\sqrt{25} = 36-25$

$= 11$

Example 6

$$\left(\sqrt{7}-\sqrt{2}\right)\left(\sqrt{7}+\sqrt{2}\right)$$

$= \sqrt{49}-\sqrt{14}+\sqrt{14}-\sqrt{4}$

the middle terms cancel

$= 7-2$

$= 5$

Rationalizing the Denominator

Rationalize the Denominator by multiplying by the binomial that will form a Conjugate with the denominator and then simplify the fraction as shown in the example on the following pages.

$\dfrac{2}{7+\sqrt{3}}$ multiply by $\dfrac{7-\sqrt{3}}{7-\sqrt{3}}$

Example 7

Simplify $\dfrac{5}{4-\sqrt{3}}$

multiply the top and bottom by $4+\sqrt{3}$

$$= \frac{5}{4-\sqrt{3}} \bullet \frac{4+\sqrt{3}}{4+\sqrt{3}}$$

Note: $\left(4-\sqrt{3}\right)\left(4+\sqrt{3}\right) = 16 - 9 = 7$

$$= \frac{5\left(4+\sqrt{3}\right)}{7} = \frac{20+5\sqrt{3}}{7}$$

Example 9

$\dfrac{9}{5-\sqrt{2}}$ multiply by $\dfrac{5+\sqrt{2}}{5+\sqrt{2}}$

Example 8

Simplify $\dfrac{7}{5+\sqrt{2}}$

multiply the top and bottom by $5-\sqrt{2}$

$$= \frac{7}{5+\sqrt{2}} \bullet \frac{5-\sqrt{2}}{5-\sqrt{2}}$$

Note: $\left(5-\sqrt{2}\right)\left(5-\sqrt{2}\right) = 25 - 2 = 23$

$$= \frac{7\left(5-\sqrt{2}\right)}{23} = \frac{35-7\sqrt{2}}{23}$$

Example 10

Simplify $\frac{8}{6+\sqrt{2}}$

multiply the top and bottom by $6-\sqrt{2}$

$$= \frac{8}{6+\sqrt{2}} \bullet \frac{6-\sqrt{2}}{6-\sqrt{2}}$$

Note: $\left(6+\sqrt{2}\right)\left(6-\sqrt{2}\right) = 36-2 = 34$

$$= \frac{8\left(6+\sqrt{2}\right)}{34}$$

the 8 outside the brackets
and the 34 are both facors
and can reduce each other

$$= \frac{\not{8}^{4}\left(6+\sqrt{2}\right)}{\not{34}^{17}}$$

$$= \frac{4\left(6+\sqrt{2}\right)}{17} = \frac{24+4\sqrt{2}}{17}$$

Example 11

Simplify $\frac{12}{5-\sqrt{3}}$

multiply the top and bottom by $5+\sqrt{3}$

$$= \frac{12}{5-\sqrt{3}} \bullet \frac{5+\sqrt{3}}{5+\sqrt{3}}$$

Note: $\left(5-\sqrt{3}\right)\left(5+\sqrt{3}\right) = 25-3 = 22$

$$= \frac{12\left(5+\sqrt{3}\right)}{22}$$

the 12 outside the brackets
and the 22 are both facors
and can reduce each other

$$= \frac{\not{12}^{6}\left(5+\sqrt{3}\right)}{\not{22}^{11}}$$

$$= \frac{6\left(5+\sqrt{3}\right)}{11} = \frac{30+6\sqrt{3}}{11}$$

Example 12

Simplify $\dfrac{6}{\sqrt{7}-\sqrt{3}}$

multiply the top and bottom by $\sqrt{7}+\sqrt{3}$

$$=\frac{6}{\sqrt{7}-\sqrt{3}}\bullet\frac{\sqrt{7}+\sqrt{3}}{\sqrt{7}+\sqrt{3}}$$

Note: $\left(\sqrt{7}-\sqrt{3}\right)\left(\sqrt{7}+\sqrt{3}\right)=7-3=4$

$$=\frac{6\left(\sqrt{7}+\sqrt{3}\right)}{4}$$

the 6 outside the brackets
and the 4 are both facors
and can reduce each other

$$=\frac{\cancel{6}^{3}\left(\sqrt{7}+\sqrt{3}\right)}{\cancel{4}^{2}}$$

$$=\frac{3\left(\sqrt{7}+\sqrt{3}\right)}{2}=\frac{3\sqrt{7}+3\sqrt{3}}{2}$$

Example 13

Simplify $\dfrac{12}{\sqrt{11}+\sqrt{2}}$

multiply the top and bottom by $\sqrt{11}-\sqrt{2}$

$$=\frac{12}{\sqrt{11}+\sqrt{2}}\bullet\frac{\sqrt{11}-\sqrt{2}}{\sqrt{11}-\sqrt{2}}$$

Note: $\left(\sqrt{11}+\sqrt{2}\right)\left(\sqrt{11}-\sqrt{2}\right)=11-2=9$

$$=\frac{12\left(\sqrt{11}-\sqrt{2}\right)}{9}$$

the 12 outside the brackets
and the 9 are both facors
and can reduce each other

$$=\frac{\cancel{12}^{4}\left(\sqrt{11}+\sqrt{2}\right)}{\cancel{9}^{3}}$$

$$=\frac{4\left(\sqrt{11}-\sqrt{2}\right)}{3}=\frac{4\sqrt{11}-4\sqrt{2}}{3}$$

Example 14

Simplify $\frac{\sqrt{3}}{4+\sqrt{2}}$

multiply the top and bottom by $4-\sqrt{2}$

$$= \frac{\sqrt{3}}{4+\sqrt{2}} \bullet \frac{4-\sqrt{2}}{4-\sqrt{2}}$$

Note: $\left(4-\sqrt{2}\right)\left(4-\sqrt{2}\right) = 16-2 = 14$

$$= \frac{\sqrt{3}\left(4-\sqrt{2}\right)}{14}$$

$$= \frac{4\sqrt{3}-\sqrt{6}}{14}$$

Simplify $\frac{\sqrt{7}}{\sqrt{5}-\sqrt{2}}$

multiply the top and bottom by $\sqrt{5}+\sqrt{2}$

$$= \frac{\sqrt{7}}{\sqrt{5}-\sqrt{2}} \bullet \frac{\sqrt{5}+\sqrt{2}}{\sqrt{5}+\sqrt{2}}$$

Note: $\left(\sqrt{5}-\sqrt{2}\right)\left(\sqrt{5}+\sqrt{2}\right) = 5-2 = 3$

$$= \frac{\sqrt{7}\left(\sqrt{5}+\sqrt{2}\right)}{3}$$

$$= \frac{\sqrt{35}+\sqrt{14}}{3}$$

Example 15

Simplify $\frac{\sqrt{2}}{3-\sqrt{6}}$

multiply the top and bottom by $3+\sqrt{6}$

$$= \frac{\sqrt{2}}{3-\sqrt{6}} \bullet \frac{3+\sqrt{6}}{3+\sqrt{6}}$$

Note: $\left(3+\sqrt{6}\right)\left(3+\sqrt{6}\right) = 9-6 = 3$

$$= \frac{\sqrt{2}\left(3+\sqrt{6}\right)}{3}$$

$$= \frac{2\sqrt{3}+\sqrt{12}}{3} = \frac{2\sqrt{3}+\sqrt{4 \bullet 3}}{3}$$

$$= \frac{2\sqrt{3}+2\sqrt{3}}{3} = \frac{2\sqrt{3}+2\sqrt{3}}{3} \quad \frac{4\sqrt{3}}{3}$$

Example 16

Simplify $\frac{\sqrt{3}}{\sqrt{15}+\sqrt{6}}$

multiply the top and bottom by $\sqrt{15}-\sqrt{6}$

$$= \frac{\sqrt{3}}{\sqrt{15}+\sqrt{6}} \bullet \frac{\sqrt{15}-\sqrt{6}}{\sqrt{15}-\sqrt{6}}$$

Note: $\left(\sqrt{15}-\sqrt{6}\right)\left(\sqrt{15}-\sqrt{6}\right) = 15-6 = 9$

$$= \frac{\sqrt{3}\left(\sqrt{15}-\sqrt{6}\right)}{9}$$

$$= \frac{\sqrt{45}+\sqrt{18}}{9} = \frac{\sqrt{9 \bullet 5}+\sqrt{9 \bullet 2}}{9}$$

$$= \frac{3\sqrt{5}+3\sqrt{2}}{9} = \frac{\sqrt{5}+\sqrt{2}}{3}$$

