Section 6 – 4B: The Inverse of a One To One Function

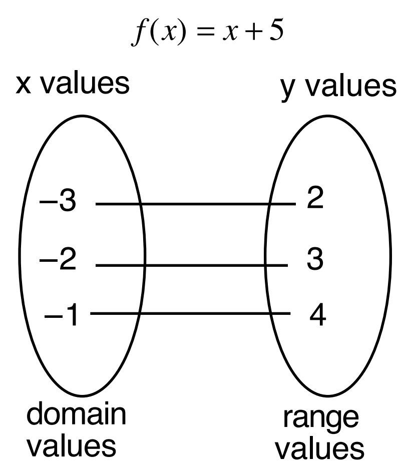
One TO One Functions

A Function is a relation that requires that **for every x value there is only one value of y related to that x. Not every relation is a function.**

Some functions are One to One Functions.

A One to One Function is a **special function** where **each x value is related to a single y value and that y value is related to only that x value.**

The function $f(x) = x + 5$ shown below is a One to One Function



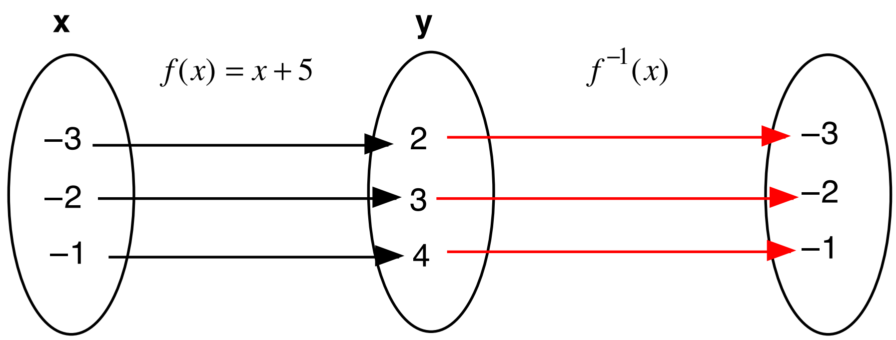
We use the variable **x for the domain values**. We use $f(x)$ or **y as the variable for the range value** that we get when we put a given x value into an equation. **For each unique x value** we put into $f(x) = x + 5$ **we get a unique y value.**

The Inverse Function $f^{-1}(x)$

$f(x) = x + 5$ is a One to One function that maps each x value to it's unique y value. Is there a function that would **"undo" or "reverse"** this? In others words, is there a function that would take each y value and map them back to their unique related x value.

If there is such a function then we say that it is the Inverse of the One to One Function $f(x)$

If there is such a function we notate it as $f^{-1}(x)$



$f(x)$ maps each x value to it's unique related y value

Is there a function $f^{-1}(x)$ that would map each y value back to it's unique related x value?

If $f(x)$ maps a given x_1 to a unique y_1

$$f(x_1) = y_1$$

then the inverse function $f^{-1}(x)$ must map the given y_1 back to it's unique x_1

$$f^{-1}(y_1) = x_1$$

If $f(x)$ maps – 3 to 2 (–3,2)	then	$f^{-1}(x)$ must map **2** to – 3 (2, –3)
If $f(x)$ maps – 2 to 3 (–2,3)	then	$f^{-1}(x)$ must map **3** to –2 (3,–2)
If $f(x)$ maps – 1 to 4 (–1,4)	then	$f^{-1}(x)$ must map **4** to – 1 (4,–1)

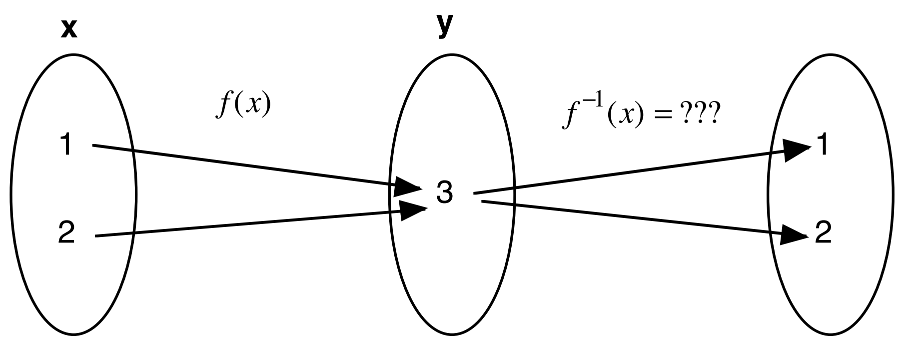
$f(x)$ must be a **One to One Function**

for the inverse function $f^{-1}(x)$ to exist

The need for $f(x)$ to be a One to One Function is made clear as you understand what an inverse function does. A function maps a unique x_1 to a unique y_1. If the inverse function $f^{-1}(x)$ "undoes" this then that unique y_1 must also have a unique x_1 to map back to. This requires that the first mapping be a One to One Function. If each x_1 value is related to a single y_1 value and that y_1 value is related to only that x_1 value then it is easy to see how $f(x)$ maps maps a unique x_1 to a unique y_1 and that the inverse function $f^{-1}(x)$ maps that unique y_1 back to it's unique x_1.

Why does $f(x)$ and $f^{-1}(x)$ have to be one to one functions?

Consider the function $f(x)$ below. $f(x)$ is a function but it is not One to One. Two values of x in the domain are mapped to one y value in the range. The inverse function $f^{-1}(x)$ can't exist. The inverse would have to map the single value of 3 back to two different different values 1 and 2. This would not meet the definition of a function so the inverse function $f^{-1}(x)$ can't exist.

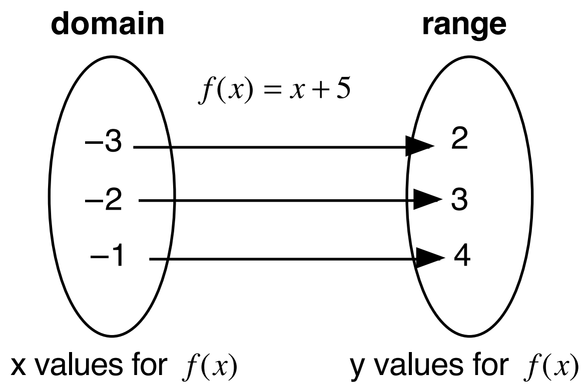


A function must be One to One for it to have an inverse.
That way each unique x value is mapped to a unique y value.
The inverse then maps each unique y value back to its respective x value.
This could only happen if the original function was one to one.

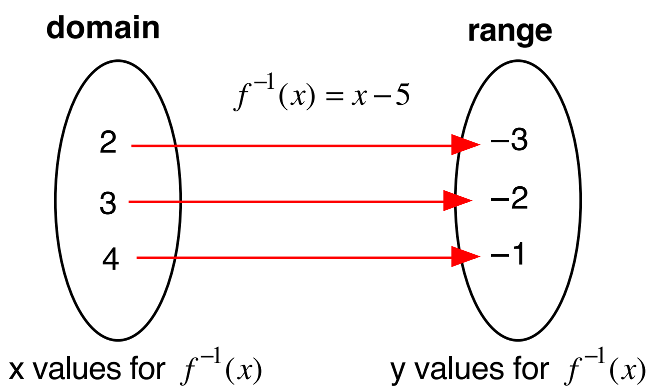
The Inverse Function $f^{-1}(x)$

Given a one to one function $f(x)$ each domain variable x is mapped to a unique range variable y. The symbol for the inverse function of $f(x)$ is written $f^{-1}(x)$.

Consider $f(x) = x + 5$ for the **domain values –3, –2, –1**
these three domain values produce **range values of 2, 3, 4**



Since $f(x) = x + 5$ is a One to One Function there must be an inverse function $f^{-1}(x)$
that takes the **range values of 2, 3, 4 for**
and maps them back to the **domain values –3, –2, –1**



The domain of the inverse function $f^{-1}(x)$ is the range of $f(x)$

and

the range for $f^{-1}(x)$ is the domain of $f(x)$

The inverse function $f^{-1}(x)$ has the effect of taking each (x, y) pair for $f(x)$ and **switching the x and y values** to get a new set of ordered pairs (y , x).

Find the inverse of a set of Ordered Pairs

Example 1

if $f(x) =$ (–2 , 4) (1 , 5) (2 , 7) (6 , 8) find $f^{-1}(x)$

If $f(x)$ maps a given x_1 to a unique y_1

then the inverse function $f^{-1}(x)$ maps the given y_1 back to it's unique x_1

To find $f^{-1}(x)$ **switch the x and y values** for each (x, y) pair in $f(x)$

$f^{-1}(x) =$ (4 , –2) (5 , 1) (7 , 2) (8 , 6)

Find the inverse of a table of Ordered Pairs

Example 2

If $f(x) =$

x	–2	–1	4	5	6
y	3	5	6	7	8

then find $f^{-1}(x)$

If $f(x)$ maps a given x_1 to a unique y_1

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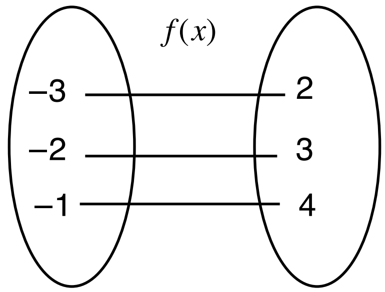
To find $f^{-1}(x)$ **switch the x and y values** for each (x, y) pair in $f(x)$

$f^{-1}(x) =$

x	3	5	6	7	8
y	–2	–1	4	5	6

Find the inverse of a mapping of Ordered Pairs.

Example 3

If $f(x) =$ 

then find $f^{-1}(x)$

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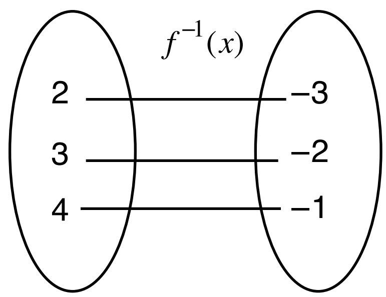
$f^{-1}(x) =$ 

Find the inverse of a graph.

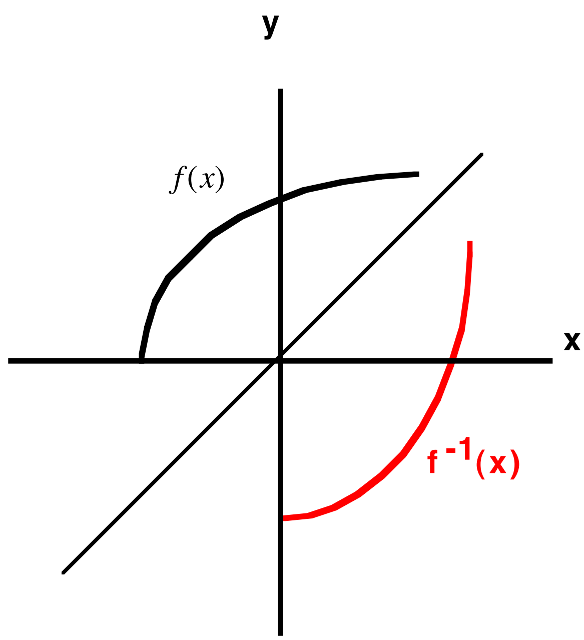
Example 4

Find the inverse of a function given it's graph

Graph the Inverse function for each graph on the same graph as the function.

Step 1: Draw the y =x line

Step 2: Reflect the f(x) line about the y = x line to find $f^{-1}(x)$



Example 5

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Graph the Inverse function for each graph on the same graph as the function.

Step 1: Draw the y =x line
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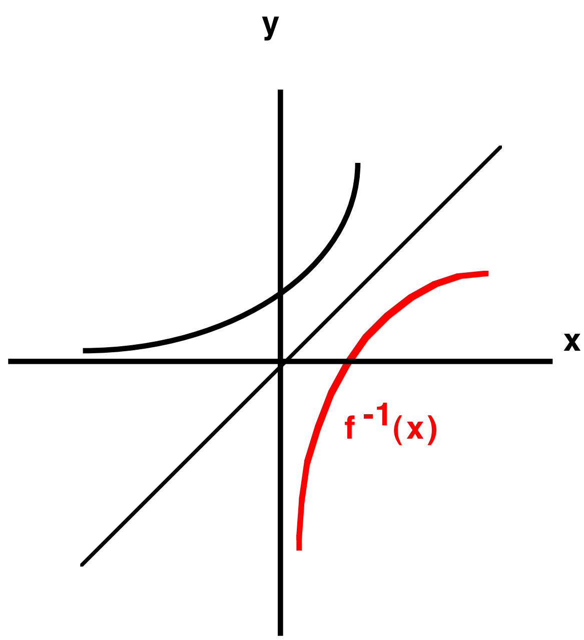

Finding the Inverse of an Equation

Step 1: Replace the $f(x)$) symbol with y

Step 2. **Switch the x and y variables**. Change the y variable into an x variable and change each x into a y variable.

Step 4. Solve for the y variable.

Step 4: Replace the y symbol with a $f^{-1}(x)$) symbol

Example 1

Find the inverse of
$f(x) = 5x - 7$

Step 1: $y = 5x - 7$

Step 2: $x = 5y - 7$

Step 3: $x = 5y - 7$

$x + 7 = 5y$

$\frac{x+7}{5} = y$

Step 4: $f^{-1}(x) = \frac{x+7}{5}$

.

Example 2

Find the inverse of
$f(x) = 6x + 2$

Step 1: $y = 6x + 2$

Step 2: $x = 6y + 2$

Step 3: $x = 6y + 2$

$x - 2 = 6y$

$\frac{x-2}{6} = y$

Step 4: $f^{-1}(x) = \frac{x-2}{6}$

Example 3

Find the inverse of

$f(x) = \frac{2}{3}x - 4$

Step 1: $y = \frac{2}{3}x - 4$

Step 2: $x = \frac{2}{3}y - 4$

Step 3: $x = \frac{2}{3}y - 4$

.

$3x = 2y - 12$

$3x + 12 = 2y$

$\frac{3x+12}{2} = y$

Step 4: $f^{-1}(x) = \frac{3x+12}{2}$

Example 4

Find the inverse of

$f(x) = \frac{-3}{4}x + 2$

Step 1: $y = \frac{-3}{4}x + 2$

Step 2: $x = \frac{-3}{4}y + 2$

Step 3: $x = \frac{-3}{4}y + 2$

$4x = -3y + 8$

$4x - 8 = -3y$

$\frac{4x-8}{-3} = y$

$\frac{-4x+8}{3} = y$

Step 4: $f^{-1}(x) = \frac{-4x+8}{3} = y$

Example 5

Find the inverse of

$f(x) = (x-2)^3$

Step 1: $y = (x-2)^3$

Step 2: $x = (y-2)^3$

take the cube root of both sides

.

Step 3: $\sqrt[3]{x} = y - 2$

$\sqrt[3]{x} + 2 = y$

Step 4: $f^{-1}(x) = \sqrt[3]{x} + 2$

Example 6

Find the inverse of

$f(x) = x^3 + 3$

Step 1: $y = x^3 + 3$

Step 2: $x = y^3 + 3$

Step 3: $x = y^3 + 3$

$x - 3 = y^3$

take the cube root of both sides

$\sqrt[3]{x-3} = y$

Step 4: $f^{-1}(x) = \sqrt[3]{x-3}$

Example 7

Find the inverse of
$f(x) = \sqrt[3]{x-4}$

Step 1: $y = \sqrt[3]{x-4}$

Step 2: $x = \sqrt[3]{y-4}$

. cube both sides

Step 3: $x^3 = y - 4$

$x^3 + 4 = y$

Step 4: $f^{-1}(x) = x^3 + 4$

Example 8

Find the inverse of
$f(x) = \sqrt[3]{x} + 2$

Step 1: $y = \sqrt[3]{x} + 2$

Step 2: $x = \sqrt[3]{y} + 2$

Step 3: $x - 2 = \sqrt[3]{y}$

cube both sides

$(x-2) = y$

Step 4: $f^{-1}(x) = (x-2)$

Example 9

Find the inverse of

$f(x)=\dfrac{7}{2x}$

Step 1: $y=\dfrac{7}{2x}$

Step 2: $x=\dfrac{7}{2y}$

Step 3: $x=\dfrac{7}{2y}$

.

multiply both side by 2y

$$2xy=7$$

divide both sides by 2x

$$y=\frac{7}{2x}$$

Step 4: $f^{-1}(x)=\dfrac{7}{2x}$

Example 10

Find the inverse of

$f(x)=\dfrac{2}{x+3}$

Step 1: $y=\dfrac{2}{x+3}$

Step 2: $x=\dfrac{2}{y+3}$

Step 3: $x=\dfrac{2}{y+3}$

multiply both side by $(y+3)$

$$(y+3)x=2$$

divide both sides by x

$$y+3=\frac{7}{x}$$

$$y=\frac{7}{x}-3$$

Step 4: $f^{-1}(x)=\dfrac{7}{x}-3$

Example 11

Find the inverse of

$f(x)=\dfrac{-3}{2x-5}$

Step 1: $y=\dfrac{-3}{2x-5}$

Step 2: $x=\dfrac{-3}{2y-5}$

Step 3: $x=\dfrac{-3}{2y-5}$

multiply both side by $(2y-5)$

$$(2y-5)x=-3$$

divide both sides by x

$$2y-5=\frac{-3}{x}$$

$$2y=\frac{-3}{x}+5$$

multiply both side by $\dfrac{1}{2}$

$$\left(\frac{1}{2}\right)2y=\left(\frac{1}{2}\right)\frac{-3}{x}+\left(\frac{1}{2}\right)5$$

$$y=\frac{-3}{2x}+\frac{5}{2}$$

Step 4: $f^{-1}(x)=y=\dfrac{-3}{2x}+\dfrac{5}{2}$

