3 – 2: Solving Systems in Three Variables

What is a system of 3 Equations with 3 Unknowns?

A system of 3 equations with 3 unknowns is a list of three equations that have 3 variables in common. Each equation can represent a plane in space. It is common to use the variables x, y, and z as the three variables to represent a plane (flat surface) in a three dimensional space. Applications in science and business use other variables that better reflect the given application. They reflect the relationship between the objects the variables represent. If a hospital wants to represent the relationship between the number of doctors, nurses and patients in three different wards they will create three equations that each use D, N and P as the variables. The variables can be any letters. This section will choose to use x, y and w as the 3 variables for all the problems in this unit.

Examples of a System of 3 Equations with 3 Unknowns

$$\begin{matrix}\text{Equation A} \\ \text{Equation B} \\ \text{Equation C}\end{matrix}\begin{cases} 2x-3y+4w=9 \\ 5x+2y-9w=2 \\ x+y-w=3 \end{cases}$$

$$\begin{matrix}\text{Equation A} \\ \text{Equation B} \\ \text{Equation C}\end{matrix}\begin{cases} 2x \quad\quad +4w=9 \\ 5x+2y \quad\quad =2 \\ \quad\quad y-w=3 \end{cases}$$

What is a solution to a System of Three Equations with Three Unknowns?

The solution to a system of three equations with three unknowns is an ordered triplet of numbers.

The order for each ordered triplet of numbers will be (x, y, w).

$(\ 3\ ,\ 5\ ,\ -2\ )$ $\quad$ $\left(-3\ ,\ 0\ ,\ \frac{1}{2}\right)$ $\quad$ $\left(3\ ,\ \frac{2}{5}\ ,\ \frac{-3}{4}\right)$

This section will choose to use x, y and w as the 3 variables for all the problems in this unit. The solution to a system of three equations with three unknowns is an ordered triplet of numbers.

The order for each ordered triplet of numbers will be (x, y, w).

An ordered triplet is a solution to a system of three equations if each equation is true when you substitute the values for each variable into all three equation. To determine if an ordered triplet is a solution the values for x y, and z will be plugged into all 3 equations for the x , y and z variables.

If the ordered triplet (x, y, w) make all 3 equations true
then the ordered triplet (x, y, w) is a solution.

If the ordered triplet (x, y, w) does not make all 3 equations true
then it is not a solution.

Is (x, y, w) **a solution to the system ?**

Example 1

Is $(1,2,3)$ a solution to

$$\begin{array}{ll} \text{Equation A} & \left\{ \begin{array}{l} 2x-3y+w=-1 \\ 5x+2y-w=6 \\ x+y-2w=-3 \end{array} \right. \\ \text{Equation B} & \\ \text{Equation C} & \end{array}$$

$(1,2,3)$ means $x=1$, $y=2$ and $w=3$

plug these values into both
Equation A , Equation B and Equation C to
see if they work in all 3 equations

$x=1$, $y=2$ and $w=3$

Equation A	$2x-3y+w=-1$	$\rightarrow$	$2(1)-3(2)+(3)=-1$
Equation B	$5x+2y-w=6$	$\rightarrow$	$5(1)+2(2)-(3)=6$
Equation C	$x+y-2w=-3$	$\rightarrow$	$(1)+(2)-2(3)=-3$

$(1,2,3)$ works in all 3 equations so
YES it is a solution.

Example 2

Is $(3,1,-1)$ a solution to

Equation A	$2x \quad + 4w = -1$
Equation B	$5x+2y \quad = 17$
Equation C	$y - w = 2$

$(3,1,-1)$ means $x=3$, $y=1$ and $w=-1$

plug these values into both
Equation A , Equation B and Equation C to
see if they work in all 3 equations

$x=3$, $y=1$ and $w=-1$

Equation A	$2x \quad + 4w = -1$	$\rightarrow$	$2(3) \quad + 4(-1) = -1$
Equation B	$5x+2y \quad = 17$	$\rightarrow$	$5(3)+2(2) \quad = 17$
Equation C	$y - w = 2$	$\rightarrow$	$(1)-1(-1)=2$

$(3,1,-1)$ works in all 3 equations so
YES it is a solution.

Is (x, y, w) **a solution to the system ?**

Example 3

Is $(2,0,-1)$ a solution to

$$\begin{array}{ll} \text{Equation A} & \left\{ \begin{array}{r} x - 4y + w = 1 \\ 2x + 3y - w = 5 \\ x + y \quad\;\; = 3 \end{array} \right. \\ \text{Equation B} & \\ \text{Equation C} & \end{array}$$

$(2,0,-1)$ means $x = 2$, $y = 0$ and $w = -1$

plug these values into both
Equation A , Equation B and Equation C to
see if they work in all 3 equations

$x = 2$, $y = 0$ and $w = -1$

$$\begin{array}{lrcl} \text{Equation A} & x - 4y + w = 1 & \rightarrow & (2) - 4(0) + (-1) = 1 \\ \text{Equation B} & 2x + 3y - w = 5 & \rightarrow & 2(2) + 3(0) - (-1) = 5 \\ \text{Equation C} & x + y \quad\;\; = 3 & \rightarrow & (2) + (0) \quad \neq 3 \end{array}$$

$(2,0,-1)$ does NOT work in equation C
No it is NOT a solution.

Solving Systems of 3 Equations with 3 Unknowns

The general procedure to solve a system of three equations with three unknowns is to use 2 elimination steps to r**educe the problem to a system of two equations with two unknowns**. You then proceed to solve the system of two equations with two unknowns just like you did in the previous section.

Steps for Solving Systems of 3 Equations with 3 Unknowns

$$\begin{array}{r} \text{Equation A} \\ \text{Equation B} \\ \text{Equation C} \end{array} \begin{cases} 2x - 3y + 3w = 12 \\ -2x + 2y - w = -6 \\ 2x + y - w = 4 \end{cases}$$

Step 1. Use a combination of **Equation A and Equation B** to eliminate one of the three variables. This will give you a new equation with two variables. Call this new equation **Equation D**

Step 2. Use a combination of **Equation B and Equation C** to eliminate the same variable as that was eliminated in step 1. This will give you a new equation with two variables. Call this new equation **Equation E**

You will now have 2 new equations. Equation C and Equation D.
Each of the new equations will have the same **two variables.**

Step 3: Use a combination of **Equation D and Equation E** to eliminate one of the two varaibles. This will allow you to solve for the variable that remains after elimination.

Step 4: Substitute the value of the variable that was solved for in step 3 into **Equation D** and solve for the variable that remains.

Step 5: Substitute the value of the variables that were solved for in step 3 and step 4 into **Equation A** and solve for the remaining variable.

You now have the values for x, y and w. State the answer as a ordered triplet.

Example 1

$$\begin{array}{r} \text{Equation A} \\ \text{Equation B} \\ \text{Equation C} \end{array}\begin{cases} 2x - 3y + 3w = 1 \\ -2x + 2y - w = -3 \\ x + y - w = 4 \end{cases}$$

Reduce the system with Three Unknowns to a new System with 2 Unknowns

We will choose to eliminate the x terms.

Step 1:
Add Equation A and Equation B
to eliminate the x terms

$A + B$ will eliminate the x terms

$$\begin{array}{r} \text{Equation A} \\ \text{Equation B} \end{array}\begin{cases} 2x - 3y + 3w = 1 \\ \underline{-2x + 2y - w = -3} \end{cases}$$

$$-y + 2w = -2$$

call the new equation Equation D

Step 2:
Add Equation B and 2 times Equation C
to eliminate the x terms

$B + 2 \bullet C$ will eliminate the x terms

$$\begin{array}{r} \text{Equation B} \\ 2 \bullet \text{Equation C} \end{array}\begin{cases} -2x + 2y - w = -3 \\ \underline{2x + 2y - 2w = 8} \end{cases}$$

$$4y - 3w = 5$$

call the new equation Equation E

New System with 2 Unknowns

$$\begin{array}{r} \text{Equation D} \\ \text{Equation E} \end{array}\begin{cases} -y + 2w = -2 \\ 4y - 3w = 8 \end{cases}$$

Step 3: Solve the System with 2 Unknowns by Elimination

We will choose to eliminate the y terms and solve for w.

$$\begin{array}{l} \text{Equation D} \\ \text{Equation E} \end{array} \begin{cases} -y+2w=-2 \\ 4y-3w=8 \end{cases}$$

$4 \bullet$ Equation D + Equation E will eliminate the y terms

$$\begin{array}{l} 4\bullet\text{Equation D} \\ \text{Equation E} \end{array} \begin{cases} -4y+8w=-8 \\ 4y-3w=8 \end{cases}$$

$2y=4$ Solve for y

$y=2$

Step 3: Plug $y=2$ into Equation D: $-y+2w=6$ and solve for w

$-(2)+2w=6$

$2w=8$

$w=4$

Step 4: Plug $y=2$ and $w=4$ into Equation A: $2x-3y+3w=12$ and solve for x

$2x-3(2)+3(4)=12$

$2x=6$

$x=3$

Answer: $(3,2,1)$

Step 5: Check:

$$\begin{array}{l} \text{Equation A} \\ \text{Equation B} \\ \text{Equation C} \end{array} \begin{cases} 2x-3y+3w=12 & 2(3)-3(2)+3(4)=12 \\ -2x+2y-w=-6 & -2(3)+2(2)-(4)=-6 \\ 2x+y-w=4 & 2(3)+(2)-(4)=4 \end{cases}$$

Solving Systems with 3 Unknowns

Example 2

$$\begin{array}{r} \text{Equation A} \\ \text{Equation B} \\ \text{Equation C} \end{array} \begin{cases} 2x + 3y + 2w = 9 \\ x - 3y - w = -4 \\ x + y - w = 0 \end{cases}$$

Step 1: Reduce the system with Three Unknowns to a new System with 2 Unknowns

We will choose to eliminate the y terms.

Add Equation A and Equation B
to eliminate the y terms
call the new equation Equation D

$$\begin{array}{r} \text{Equation A} \\ \text{Equation B} \end{array} \begin{cases} 2x + 3y + 2w = 9 \\ \underline{x - 3y - w = -4} \end{cases}$$

$$3x + w = 5$$

Add Equation B and 3• Equation C
to eliminate the y terms
call the new equation Equation E

$$\begin{array}{r} \text{Equation B} \\ 3 \bullet \text{Equation C} \end{array} \begin{cases} x - 3y - w = -4 \\ \underline{3x + 3y - 3w = 0} \end{cases}$$

$$4x - 4w = -4$$

New System with 2 Unknowns

$$\begin{array}{r} \text{Equation D} \\ \text{Equation E} \end{array} \begin{cases} 3x + w = 5 \\ 4x - 4w = -4 \end{cases}$$

Step 2: Solve the System with 2 Unknowns by Elimination

We will choose to eliminate the w terms and solve for y.

$$\begin{aligned}&\text{Equation D}\\&\text{Equation E}\end{aligned}\begin{cases}3x + w = 5\\4x - 4w = -4\end{cases}$$

Add 4 • Equation D and Equation E to eliminate the w terms

$$\begin{aligned}&4 \bullet \text{Equation D}\\&\text{Equation E}\end{aligned}\begin{cases}12x + 4w = 20\\\underline{4x - 4w = -4}\end{cases}$$

$16x = 16$ Solve for 1

$x = 1$

Step 3: Plug $x = 1$ into Equation D: $3x + w = 5$ and solve for w

$$3(1) + w = 5$$
$$3 + w = 5$$
$$w = 2$$

Step 4: Plug $x = 1$ and $w = 2$ into Equation A: $2x + 3y + 2w = 9$ and solve for y

$$2(1) + 3y + 2(2) = 9$$
$$3y = 3$$
$$y = 1$$

Answer: (1 ,1 ,2)

Step 5: Check:

$$\begin{aligned}&\text{Equation A}\\&\text{Equation B}\\&\text{Equation C}\end{aligned}\begin{cases}2x + 3y + 2w = 9 & 2(1) + 3(1) + 2(2) = 9\\x - 3y - w = -4 & (1) - 3(1) - (2) = -4\\x + y - w = 0 & (1) + (1) - (2) = 0\end{cases}$$

Solving Systems with 3 Unknowns

Example 3

$$\begin{array}{l} \text{Equation A} \\ \text{Equation B} \\ \text{Equation C} \end{array} \begin{cases} 3x - y = -7 \\ 3x - 2y + 2w = -4 \\ 2x - 3y + w = -5 \end{cases}$$

Step 1: Reduce the system with Three Unknowns to a new System with 2 Unknowns

We will choose to eliminate the w terms since Equation A has no w terms.

Equation A has no w terms
so we use it as is
and call it equation Equation D

Equation A: $3x - y = -7$

Add Equation B and $-2 \bullet$ Equation C
to eliminate the w terms
call the new equation Equation E

$$\begin{array}{r} \text{Equation B} \\ -2 \bullet \text{Equation C} \end{array} \begin{cases} 3x - 2y + 2w = -4 \\ \underline{-4x + 6y - 2w = 10} \end{cases}$$

$$-x + 4y = 6$$

New System with 2 Unknowns

$$\begin{array}{l} \text{Equation D} \\ \text{Equation E} \end{array} \begin{cases} 3x - y = -7 \\ -x + 4y = 6 \end{cases}$$

Step 2: Solve the New System with 2 Unknowns by Elimination

We will choose to eliminate the y terms and solve for x.

$$\begin{array}{l} \text{Equation D} \\ \text{Equation E} \end{array} \begin{cases} 3x - y = -7 \\ -x + 4y = 6 \end{cases}$$

Add 4 • Equation D and Equation E to eliminate the w terms

$$\begin{array}{r} 4 \bullet \text{Equation D} \\ \text{Equation E} \end{array} \begin{cases} 12x - y = -28 \\ \underline{-x + 4y = 6} \end{cases}$$

$11x = -22$ Solve for x

$x = -2$

Step 3: Plug $x = -2$ into Equation D: $3x - y = -7$ and solve for y

$$3(-2) - y = -7$$
$$-y = -1$$
$$y = 1$$

Step 4: Plug $x = -2$ and $y = 1$ into Equation B: $3x - 2y + 2w = -4$ and solve for w

$$3(-2) - 2(1) + 2w = -4$$
$$2w = 4$$
$$w = 2$$

Answer: $(-2\ ,1\ ,2)$

Step 5: Check:

$$\begin{array}{l} \text{Equation A} \\ \text{Equation B} \\ \text{Equation C} \end{array} \begin{cases} 3x - y = -7 \\ 3x - 2y + 2w = -4 \\ 2x - 3y + w = -5 \end{cases} \quad \begin{array}{r} 3(-2) - (1) = -7 \\ 3(-2) - 2(1) + 2(2) = -4 \\ 2(-2) - 3(1) + (2) = -5 \end{array}$$

The Intersection of a system of two equations with two variables

The Intersection of Two Lines on a Plane
has 3 Possible Cases

Two linear equations with two variables x and y can be modeled as the equations of two lines on a plane. There are three different geometric descriptions for the three possible ways that two lines on a plane can meet. Each different geometric description has a different type of solution for the system of two linear equations.

1. If the two lines intersect then they have a single point in common. That single point is the only ordered pair (x,y) that makes both of the equations true. The only solution to this system is the ordered pair (x , y) where the two lines intercept.

2. If the two lines have the same slope but a different y intercepts then the two lines are parallel. Parallel lines do not intersect so they have no points in common. A system of two parallel lines have no solution.

3. If one equation for the line are a multiple of the other equation then the two lines have the same slope but a different y intercept. Both equations represent the same line. All the points on one line are also on the other line because the equations represent the same line. Either linear equation can represent the solution to the system as they are the same equation but in a different form.

The Intersection of a system of three equation with three variables.

A similar geometric description apples to a system of three equation with three variables. An equation with the variables (x , y, z) can be modeled as a plane or a flat surface that extends without end. A system of three equation with three variables represents three planes. There are three general statements that describe possible arraignments of three planes in space. The three planes can meet at a single point. The three planes can have no points in common to all three planes. The three planes can meet at an infinite number of points.

The geometric drawings of these types of systems is not as simple as is was with two lines. The only way two lines on a plane can have no points in common is if the two lines are parallel. This concept cannot be extended to three planes in space. The three planes may all be may be parallel to each other but other graphical arraignments are also possible that produce three planes that do not have a point common to all three planes. It can be difficult to determine the exact arraignment of the three planes without graphing them. Graphing three planes in a three dimensional system by hand can be difficult, It is common to use a computer software program to do this.

The Intersection of Three Planes in Space
has Several Possible Cases

1. Three planes can intersect so that they have a one point in common. That single point in space is the only ordered triplet (x , y , z) that makes all three of the equations true. This unit will only deal with the case where the three planes meet at one common point.

Future courses will develop the solutions to these other cases in detail.

2. Three planes can be placed so that they three different planes that are parallel to each other. Parallel planes do not intersect so they have no points in common. A system of parallel planes have no solution.

3. Three planes can be placed so that any two of the three planes intersect each other but all three planes do not intersect each other in a common point. The planes do not have a point common to all three planes. Such a system of three planes have no solution.

4. Three planes can be placed so that all three planes intersect at a common line. Any point (x, y, z) on the line of intersection will make al three equations true. The linear equation of the intersection line represents the solution to the system.

5. If each equation for the three planes are a multiple of the other equation then the three equations represent the same plane. All the points on one plane are also on the other plane. All the points on one plane are also on the other planes because the other equations represent the same plane. Any of the three equations can represent the solution to the system as they are the same equation but in a different form.

