Section 5 – 3: Solving a System of Equations by Elimination

The Addition Property of Equality states that

You can add the same **number** to both sides of an equation and still have an equivalent equation.

If $A = B$ then
you can add the same number
C to both sides of the equation

$$A = B$$
$$+C \quad +C$$

results *in*

$$A + C = B + C$$

The Addition Property of Equality can be expanded to say that you can add
equal quantities
to both sides of an equation and still have an equivalent equation.

The Addition Property of Equality

If $A = B$ then
you can add the same quanity
C to both sides of the equation

$$A = B$$

and

$$C = D$$

then
you can add C to the left side
and D to the right side
because C and D are equal to each other
so

$$A + C = B + D$$

This property allows you to add two separate equations together and get one equivalent equation. The solution to the new equation must also be a solution to the original equations. This property is useful with some systems of equations where adding the two equations together will eliminate one of the two variables and leave you with an equation with one variable. This new equation can then be solved for that variable.

$$\begin{array}{l} \text{Equation A} \\ \text{Equation B} \end{array} \begin{cases} 3x - 5y = 3 \\ \underline{1x + 5y = 9} \end{cases}$$
$$4x \qquad = 12$$

$$\begin{array}{l} \text{Equation A} \\ \text{Equation B} \end{array} \begin{cases} 3x + 5y = 2 \\ \underline{-3x + y = -14} \end{cases}$$
$$6y = -12$$

Solve each system by the elimination method.

Example 1

If you add the left sides of Equation A and Equation B together and add the right sides of Equation A and Equation B together the x terms add to zero and you will have eliminated the x terms. You now have a new equation with only the y variable.

$$\begin{array}{r} \text{Equation A} \\ \text{Equation B} \end{array} \begin{cases} 3x + 5y = 2 \\ -3x + y = -14 \end{cases}$$

Add Equation A and Equation B to eliminate the x terms

$$\begin{array}{r} 3x + 5y = 2 \\ \underline{-3x + y = -14} \\ 6y = -12 \\ y = -2 \end{array} \quad \text{Solve for y}$$

Plug $y = -2$ into either equation A or B and solve for x

Equation A

$3x + 5(-2) = 2$

$3x - 10 = 2$

$3x = 12$

$x = 4$

Answer: (4,–2)

check:

$$\begin{cases} 3(4) + 5(-2) = 2 \\ -3(4) + (-2) = -14 \end{cases}$$

$$\begin{cases} 12 - 10) = 2 \\ -12 - 2 = -14 \end{cases}$$

Example 2

If you add the left sides of Equation A and Equation B together and add the right sides of Equation A and Equation B together the y terms add to zero and you will have eliminated the y terms. You now have a new equation with only the x variable.

$$\begin{array}{r} \text{Equation A} \\ \text{Equation B} \end{array} \begin{cases} 3x - 2y = 3 \\ 5x + 2y = 30 \end{cases}$$

Add Equation A and Equation B to eliminate the y terms

$$\begin{array}{r} 3x - 2y = 2 \\ \underline{5x + 2y = 30} \\ 8x \quad = 32 \\ x = 4 \end{array} \quad \text{Solve for x}$$

Plug $x = 4$ into either equation A or B and solve for y

Equation B

$5(4) + 2y = 30$

$20 + 2y = 30$

$2y = 10$

$y = 5$

Answer: (4,5)

check:

$$\begin{cases} 3(4) - 2(5) = 2 \\ 5(4) + 2(5) = 30 \end{cases}$$

$$\begin{cases} 12 - 10 = 2 \\ 20 + 10 = 30 \end{cases}$$

Example 3

If you add the left sides of
Equation A and Equation B together
and add the right sides of
Equation A and Equation B together
the x terms add to zero and you will
have eliminated the x terms. You now
have a new equation with only the y variable.

$$\begin{array}{r} \text{Equation A} \\ \text{Equation B} \end{array}\begin{cases} 5x-4y=3 \\ -5x+y=3 \end{cases}$$

Add Equation A and Equation B
to eliminate the x terms

$$\begin{array}{r} 5x-4y=3 \\ \underline{-5x+y=3} \\ -3y=6 \\ y=-2 \end{array} \quad \text{Solve for y}$$

Plug $y=-2$ into either equation A or B
and solve for x

Equation A

$$\begin{array}{c} 5x-4(-2)=3 \\ 5x+8=3 \\ 5x=-5 \\ x=-1 \end{array}$$

Answer: $(-1,-2)$

check:

$$\begin{cases} 5(-1)-4(-2)=3 \\ -5(-1)+(-2)=3 \end{cases}$$

$$\begin{cases} -5+8=3 \\ 5-2=3 \end{cases}$$

Example 4

If you add the left sides of
Equation A and Equation B together
and add the right sides of
Equation A and Equation B together
the y terms add to zero and you will
have eliminated the y terms. You now
have a new equation with only the x variable.

$$\begin{array}{r} \text{Equation A} \\ \text{Equation B} \end{array}\begin{cases} 3x-y=7 \\ 2x+y=3 \end{cases}$$

Add Equation A and Equation B
to eliminate the y terms

$$\begin{array}{r} 3x-y=7 \\ \underline{2x+y=3} \\ 5x\quad=10 \\ x=2 \end{array} \quad \text{Solve for x}$$

Plug $x=2$ into either equation A or B
and solve for y

Equation B

$$\begin{array}{c} 2(2)+y=3 \\ 4+y=3 \\ y=-1 \end{array}$$

Answer: $(2,-1)$

check:

$$\begin{cases} 3(2)-(-1)=7 \\ 2(2)+(-1)=3 \end{cases}$$

$$\begin{cases} 6+1=7 \\ 4-1=3 \end{cases}$$

Special Cases: No Solution or All points On The Line

Example 5

$$\begin{array}{l} \text{Equation A} \\ \text{Equation B} \end{array}\begin{cases} 2x+7y=3 \\ -2x-7y=5 \end{cases}$$

Add Equation A and Equation B

$$\begin{array}{r} 2x+7y=3 \\ \underline{-2x-7y=5} \\ 0=8 \end{array}$$

Stop: Both the x and y terms canceled out and the remaining equation $0=8$ is false

The lines are parallel, they have no common points

Answer: No Solution

Example 6

$$\begin{array}{l} \text{Equation A} \\ \text{Equation B} \end{array}\begin{cases} -4x-5y=3 \\ 4x+5y=-3 \end{cases}$$

Add Equation A and Equation B

$$\begin{array}{r} -4x-5y=3 \\ \underline{4x+5y=-3} \\ 0=0 \end{array}$$

Stop: Both the x and y terms canceled out and the remaining equation $0=0$ is true

Both equations describe the same line any point on $-4x-5y=3$ would also be on $4x+5y=-3$

Answer: All Points on $4x+5y=-3$

or

Answer: All Points on $4x-5y=3$

either one of the above is correct

What if adding Equation A and Equation B together does not eliminate one of the Variables?

Adding Equation A and Equation B together to eliminate one variable worked in each of the examples above. This procedure will only work if the coefficients of both x terms are the same number but different signs or if the coefficients of both y terms are the same number with different signs.

the coefficients of both x terms are the same number but with different signs $-7x$ and $7x$ so the x terms add to zero and will be eliminated

$$\begin{array}{l} \text{Equation A} \\ \text{Equation B} \end{array} \begin{cases} -7x+5y=2 \\ 7x+y=-14 \end{cases}$$

the coefficients of both y terms are the same number but with different signs $9y$ and $-9y$ so the y terms add to zero and will be eliminated

$$\begin{array}{l} \text{Equation A} \\ \text{Equation B} \end{array} \begin{cases} 3x+9y=3 \\ 8x-9y=30 \end{cases}$$

The Multiplication Property of Equality

You can multiply both sides of an equation by the same number and still have an equivalent equation.

If the coefficients of the x or y terms **are the not the same number with different signs then the addition of the two equations will not eliminate one of the two variables.** You can use the **Multiplication Property of Equality** to transform the two equations so that the coefficients of either the x or y terms add to zero. This step requires that you **multiply one or both of the equations by a number that will cause either the x or y terms add to zero.**

Example 7

$$\begin{array}{l} \text{Equation A} \\ \text{Equation B} \end{array} \begin{cases} 4x+3y=16 \\ -2x+y=2 \end{cases}$$

multiply Equation B by 2 so that Equation A has 4x and Equation B has $-4x$

$$\begin{array}{l} \\ 2 \end{array} \begin{cases} 4x+3y=16 \\ -2x+y=2 \end{cases}$$

$$\begin{array}{r} 4x+3y=16 \\ \underline{-4x+2y=4} \end{array}$$

this will eliminate the x terms

Example 8

$$\begin{array}{l} \text{Equation A} \\ \text{Equation B} \end{array} \begin{cases} 4x+3y=5 \\ 5x+3y=2 \end{cases}$$

multiply Equation A by -1 so that Equation A has a $-3y$ and Equation B has a $3y$

$$\begin{array}{l} -1 \\ \\ \end{array} \begin{cases} 4x+3y=5 \\ 5x+3y=2 \end{cases}$$

$$\begin{array}{r} -4x-3y=-5 \\ \underline{8x+3y=2} \end{array}$$

this will eliminate the y terms

Sometimes you can eliminate either the x terms or the y terms.
It's your choice which one you decide to eliminate.

Example 9A shows a system where the **x terms are eliminated** by multiplying Equation A by –1. This eliminates the x terms and **leaves an equation in terms of y.**

Example 9A

$$\begin{array}{r} \text{Equation A} \\ \text{Equation B} \end{array} \begin{cases} -2x+6y=5 \\ -2x+3y=11 \end{cases}$$

multiply Equation A by -1 so that
Equation A has 2x and
Equation B has $-2x$

$$-1\begin{cases} -2x+6y=5 \\ -2x+3y=2 \end{cases}$$

$$\begin{array}{l} 2x-6y=-5 \\ \underline{-2x+3y=11} \end{array}$$

this will eliminate the x terms

Example 9B shows the **same system** but the **y terms are eliminated** by multiplying Equation B by –2. This eliminates the y terms and **leaves an equation in terms of x.**

Example 9B

$$\begin{array}{r} \text{Equation A} \\ \text{Equation B} \end{array} \begin{cases} -2x+6y=5 \\ -2x+3y=11 \end{cases}$$

multiply Equation B by -2 so that
Equation A has 6y and
Equation B has $-6y$

$$\begin{array}{r} \\ -2 \end{array}\begin{cases} -2x+6y=5 \\ -2x+3y=11 \end{cases}$$

$$\begin{array}{l} -2x+6y=5 \\ \underline{4x-6y=-22} \end{array}$$

this will eliminate the y terms

Example 10

Equation A, Equation B $\begin{cases} 4x+3y=16 \\ -2x+y=2 \end{cases}$

Multiply Equation B by 2
to eliminate the x terms

$$\begin{matrix} \\ 2 \end{matrix}\begin{cases} 4x+3y=16 \\ -2x+y=2 \end{cases}$$

$$\begin{array}{r} 4x+3y=16 \\ \underline{-4x+2y=4} \\ 5y=20 \\ y=4 \end{array}$$

Now add the two equations

Solve for y

Plug $y=4$ into either equation A or B
and solve for x

Equation B
$4x+3(4)=16$
$4x+12=16$
$4x=4$
$x=1$
Answer: (1,4)

check:

$$\begin{cases} 4(1)+3(4)=16 \\ -2(1)+(4)=2 \end{cases} \qquad \begin{cases} 4x+3y=16 \\ -2x+y=2 \end{cases}$$

$$\begin{cases} 4+12=16 \\ -2+4=2 \end{cases}$$

Example 11

Equation A, Equation B $\begin{cases} 3x+y=2 \\ 2x+3y=20 \end{cases}$

Multiply Equation A by -3
to eliminate the y terms

$$\begin{matrix} -3 \\ \\ \end{matrix}\begin{cases} 3x+y=2 \\ 2x+y=2 \end{cases}$$

$$\begin{array}{r} -9x-3y=-6 \\ \underline{2x+3y=20} \\ -7x=14 \\ x=-2 \end{array}$$

Now add the two equations

Solve for x

Plug $x=-2$ into either equation A or B
and solve for y

Equation A
$3(-2)+y=2$
$-6+y=2$
$y=8$

Answer: (–2,8)

check:

$$\begin{cases} 3(-2)+(8)=2 \\ 2(-2)+3(8)=20 \end{cases} \qquad \begin{cases} 3x+y=2 \\ 2x+3y=20 \end{cases}$$

$$\begin{cases} -6+8=2 \\ -4+24=20 \end{cases}$$

Example 12

$$\begin{cases} \text{Equation A} & 2x+2y=7 \\ \text{Equation B} & 4x-3y=-7 \end{cases}$$

Multiply Equation A by -2
to eliminate the x terms

$$\begin{matrix} -2 \\ \end{matrix}\begin{cases} 2x+2y=7 \\ 4x-3y=-7 \end{cases}$$

$$\begin{array}{rl} -4x-4y=-14 & \\ \underline{4x-3y=-7} & \text{Add the two equations} \\ -7y=-21 & \text{Solve for y} \\ y=3 & \end{array}$$

Plug $y=3$ into either equation A or B
and solve for x

Equation B

$4x-3(3)=-7$

$4x-9=-7$

$4x=2$

$x=\frac{1}{2}$

Answer: $\left(\frac{1}{2},3\right)$

check:

$$\begin{cases} 2\left(\frac{1}{2}\right)+2(3)=7 \\ 4\left(\frac{1}{2}\right)-3(3)=-7 \end{cases} \qquad \begin{cases} 2x+2y=7 \\ 4x-3y=-7 \end{cases}$$

Example 13

$$\begin{cases} \text{Equation A} & 4x-6y=-2 \\ \text{Equation B} & 2x+3y=3 \end{cases}$$

Multiply Equation B by 2
to eliminate the y terms

$$\begin{matrix} \\ 2 \end{matrix}\begin{cases} 4x-6y=-2 \\ 2x+3y=3 \end{cases}$$

$$\begin{array}{rl} 4x-6y=-2 & \\ \underline{4x+6y=\ \ 6} & \text{Add the two equations} \\ 8x\ \ \ \ \ =4 & \text{Solve for x} \end{array}$$

$$x=\frac{4}{8}=\frac{1}{2}$$

Plug $x=\frac{1}{2}$ into either equation A or B
and solve for y

Equation A

$4\left(\frac{1}{2}\right)-6y=-2$

$2-6y=-2$

$-6y=-4$

$y=\frac{-4}{-6}=\frac{2}{3}$

Answer: $\left(\frac{1}{2},\frac{2}{3}\right)$

check:

$$\begin{cases} 4\left(\frac{1}{2}\right)-6\left(\frac{2}{3}\right))=-2 \\ 2\left(\frac{1}{2}\right)+3\left(\frac{2}{3}\right)=3 \end{cases} \qquad \begin{cases} 4x-6y=-2 \\ 2x+3y=3 \end{cases}$$

You may need to multiply both equations by different numbers to eliminate one of the variables

Example 14A

We will eliminate the x variables

Equation A $\begin{cases} 2x-3y=3 \\ 3x+4y=13 \end{cases}$ Equation B

You must mutiply both rows by different numbers to eliminate a variable

Multiply Equation A by -3
Multiply Equation B by 2
to eliminate the x terms

$$\begin{matrix} -3 \\ 2 \end{matrix}\begin{cases} 2x-3y=3 \\ 3x+4y=13 \end{cases}$$

$$\begin{array}{r} -6x+9y=-9 \\ \underline{6x+8y=26} \\ 17y=17 \\ y=1 \end{array}$$

Now add the two equations

Solve for y

Plug $y=1$ into either equation A or B and solve for x

Equation B
$3x+4(1)=13$
$3x+4=13$
$3x=9$
$x=3$
Answer: $(3,1)$

check:

$\begin{cases} 2(3)-3(1)=3 \\ 3(3)+4(1)=13 \end{cases}$ $\begin{cases} 2x-3y=3 \\ 3x+4y=13 \end{cases}$

Example 14B

This is the **same problem as Example 14A** but we will eliminate the y variables

Equation A $\begin{cases} 2x-3y=3 \\ 3x+4y=13 \end{cases}$ Equation B

You must mutiply both rows by different numbers to eliminate a variable

Multiply Equation A by 4
Multiply Equation B by 3
to eliminate the y terms

$$\begin{matrix} 4 \\ 3 \end{matrix}\begin{cases} 2x-3y=3 \\ 3x+4y=13 \end{cases}$$

$$\begin{array}{l} 8x-12y=12 \\ \underline{9x+12y=39} \\ 17x \qquad =51 \\ \qquad x=3 \end{array}$$

Now add the two equations

Solve for x

Plug $x=3$ into either equation A or B and solve for y

Equation A
$2(3)-3y=3$
$6-3y=3$
$-3y=-3$
$y=1$
Answer: $(3,1)$

check:

$\begin{cases} 2(3)-3(1)=3 \\ 3(3)+4(1)=13 \end{cases}$ $\begin{cases} 2x-3y=3 \\ 3x+4y=13 \end{cases}$

Example 15A

We will eliminate the x variables

Equation A / Equation B
$$\begin{cases} -3x+5y=9 \\ 4x+2y=14 \end{cases}$$

You must mutiply both rows by different numbers to eliminate a variable

Multiply Equation A by 4
Multiply Equation B by 3
to eliminate the x terms

$$\begin{matrix} 4 \\ 3 \end{matrix}\begin{cases} -3x+5y=9 \\ 4x+2y=14 \end{cases}$$

$$\begin{aligned} -12x+20y&=36 \\ 12x+6y&=42 \quad \text{Now add the two equations} \\ 26y&=78 \quad \text{Solve for y} \\ y&=3 \end{aligned}$$

Plug $y=3$ into either equation A or B and solve for x

Equation A
$-3x+5(3)=9$
$-3x+15=9$
$-3x=-6$
$x=2$
Answer: $(2,3)$

check:
$$\begin{cases} -3(2)+5(3)=9 \\ 4(2)+2(3)=14 \end{cases} \qquad \begin{cases} -3x+5y=9 \\ 4x+2y=14 \end{cases}$$

Example 15B

This is the **same problem as Example 15B** but we will eliminate the y variables

Equation A / Equation B
$$\begin{cases} -3x+5y=9 \\ 4x+2y=14 \end{cases}$$

You must mutiply both rows by different numbers to eliminate a variable

Multiply Equation A by -2
Multiply Equation B by 5
to eliminate the y terms

$$\begin{matrix} -2 \\ 5 \end{matrix}\begin{cases} -3x+5y=9 \\ 4x+2y=14 \end{cases}$$

$$\begin{aligned} 6x-10y&=-18 \\ 20x+10y&=70 \quad \text{Now add the two equations} \\ 26x&=52 \quad \text{Solve for x} \\ x&=2 \end{aligned}$$

Plug $y=2$ into either equation A or B and solve for x

Equation A
$-3(2)+5y=9$
$-6+5y=9$
$5y=15$
$y=3$
Answer: $(2,3)$

check:
$$\begin{cases} -3(2)+5(3)=9 \\ 4(2)+2(3)=14 \end{cases} \qquad \begin{cases} -3x+5y=9 \\ 4x+2y=14 \end{cases}$$

Note: Example 11 and 12 solved the exact same system two different ways. Example 11 eliminated the x variables first and example 12 solved the same system again by eliminating the y variables. You will never be asked to solve a system both ways as we did above. This was done so you could see it does not matter which variable you chose to eliminate.

What if the system has fractions in the equations?

Eliminate the fractions and get a system without fractions.

Systems with fractions can look overwhelming at first. The key to making these systems easier is to multiply each equation by the LCD (least common denominator) for the fractions in that equation. This will eliminate the denominators and give you a new system with out fraction. Solve this new system like the examples above.

Note: The examples below will not be completely solved. The steps to eliminate the fractions and get a new system without fractions will be shown. The remaining steps to solve the system are left to the student.

Example 16

$$\begin{cases} \text{Equation A} & \dfrac{x}{3} - \dfrac{y}{6} = \dfrac{-2}{3} \\ \text{Equation B} & \dfrac{-x}{2} - \dfrac{y}{4} = 1 \end{cases}$$

First multiply each equation by the LCD for that equation The resulting equations will not contain any fractions.

Multiply Equation A by 6
Multiply Equation B by 4

$$\begin{cases} \text{Equation A} & \dfrac{6}{1}\left(\dfrac{x}{3}\right) - \dfrac{6}{1}\left(\dfrac{y}{6}\right) = \dfrac{6}{1}\left(\dfrac{-2}{3}\right) \\ \text{Equation B} & \dfrac{4}{1}\left(\dfrac{-x}{2}\right) - \dfrac{4}{1}\left(\dfrac{y}{4}\right) = \dfrac{4}{1} \bullet (1) \end{cases}$$

$$\begin{cases} \text{Equation A} & 2x - y = -4 \\ \text{Equation B} & -2x - y = 4 \end{cases}$$

Now you can solve this system like the previous examples.

Example 17

$$\begin{cases} \text{Equation A} & \dfrac{3x}{4} - \dfrac{y}{6} = \dfrac{-3}{8} \\ \text{Equation B} & \dfrac{x}{6} + \dfrac{3y}{4} = 2 \end{cases}$$

First multiply each equation by the LCD for that equation The resulting equations will not contain any fractions.

Multiply Equation A by 24
Multiply Equation B by 12

$$\begin{cases} \text{Equation A} & \dfrac{24}{1}\left(\dfrac{3x}{4}\right) - \dfrac{24}{1}\left(\dfrac{y}{6}\right) = \dfrac{24}{1}\left(\dfrac{-3}{8}\right) \\ \text{Equation B} & \dfrac{12}{1}\left(\dfrac{x}{6}\right) + \dfrac{12}{1}\left(\dfrac{3y}{4}\right) = \dfrac{12}{1} \bullet (2) \end{cases}$$

$$\begin{cases} \text{Equation A} & 18x - 4y = -9 \\ \text{Equation B} & 2x + 9y = 24 \end{cases}$$

Now you can solve this system like the previous examples.

