Section 5 – 3: Solving a System of Equations by Elimination

The Addition Property of Equality states that

You can add the same number to both sides of an equation and still have an equivalent equation.

\[
\text{If } A = B \text{ then} \\
\text{you can add the same number} \\
\text{C to both sides of the equation} \\
A = B \\
+C + C
\]

\[
\text{this results in} \\
A + C = B + C
\]

The Addition Property of Equality can be expanded to say that you can add equal quantities to both sides of an equation and still have an equivalent equation.

The Addition Property of Equality

\[
\text{If } A = B \text{ and} \\
C = D \\
\text{then} \\
\text{you can add C to the left side} \\
\text{and D to the right side} \\
because C and D are equal to each other \\
\text{so} \\
A + C = B + D
\]

This property allows you to add two separate equations together and get one equivalent equation. The solution to the new equation will also be a solution to the original equations. This property is useful with systems of equations where adding the two equations together will eliminate one of the two variables and leave you with an equation with one variable. This new equation can then be solved for that variable.
Solve each system by the elimination method.

**Example 1**

If you add the left sides of Equation A and Equation B together and add the right sides of Equation A and Equation B together the x terms add to zero and you will have eliminated the x terms. You now have a new equation with only the y variable.

\[
\begin{align*}
\text{Equation A} & : & 3x + 5y &= 2 \\
\text{Equation B} & : & -3x + y &= -14
\end{align*}
\]

Add Equation A and Equation B to eliminate the x terms

\[
\begin{align*}
3x + 5y &= 2 \\
-3x + y &= -14 \\
\hline
6y &= -12 \\
y &= -2
\end{align*}
\]

Plug \(y = -2\) into either equation A or B and solve for x

**Equation A**

\[
\begin{align*}
3x + 5(-2) &= 2 \\
3x - 10 &= 2 \\
3x &= 12 \\
x &= 4
\end{align*}
\]

Answer: \((4, -2)\)

**check:**

\[
\begin{align*}
\begin{cases}
3(4) + 5(-2) &= 2 \\
-3(4) + (-2) &= -14
\end{cases}
\end{align*}
\]

**Example 2**

If you add the left sides of Equation A and Equation B together and add the right sides of Equation A and Equation B together the y terms add to zero and you will have eliminated the y terms. You now have a new equation with only the x variable.

\[
\begin{align*}
\text{Equation A} & : & 3x - 2y &= 2 \\
\text{Equation B} & : & 5x + 2y &= 30
\end{align*}
\]

Add Equation A and Equation B to eliminate the y terms

\[
\begin{align*}
3x - 2y &= 2 \\
5x + 2y &= 30 \\
\hline
8x &= 32 \\
x &= 4
\end{align*}
\]

Plug \(x = 4\) into either equation A or B and solve for y

**Equation A**

\[
\begin{align*}
5(4) + 2y &= 30 \\
20 + 2y &= 30 \\
2y &= 10 \\
y &= 5
\end{align*}
\]

Answer: \((4, 5)\)

**check:**

\[
\begin{align*}
\begin{cases}
3(4) - 2(5) &= 2 \\
5(4) + 2(5) &= 30
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
12 - 10 &= 2 \\
20 + 10 &= 30
\end{cases}
\end{align*}
\]
Example 3

If you add the left sides of
Equation A and Equation B together
and add the right sides of
Equation A and Equation B together
the x terms add to zero and you will
have eliminated the x terms. You now
have a new equation with only the y variable.

Equation A
\[ 5x - 4y = 3 \]
Equation B
\[ -5x + y = 3 \]

Add Equation A and Equation B
to eliminate the x terms

\[
\begin{align*}
5x - 4y &= 3 \\
-5x + y &= 3 \\
\hline
-3y &= 6 \\
\end{align*}
\]
Solve for y
\[ y = -2 \]

Plug \( y = -2 \) into either equation A or B
and solve for x

Equation A
\[
\begin{align*}
5x - 4(-2) &= 3 \\
5x + 8 &= 3 \\
5x &= -5 \\
x &= -1
\end{align*}
\]
Answer: \((-1, -2)\)

check:
\[
\begin{align*}
5(-1) - 4(-2) &= 3 \\
-5(-1) + (-2) &= 3 \\
-5 + 8 &= 3 \\
5 - 2 &= 3
\end{align*}
\]

Example 4

If you add the left sides of
Equation A and Equation B together
and add the right sides of
Equation A and Equation B together
the y terms add to zero and you will
have eliminated the y terms. You now
have a new equation with only the x variable.

Equation A
\[ 3x - y = 7 \]
Equation B
\[ 2x + y = 3 \]

Add Equation A and Equation B
to eliminate the y terms

\[
\begin{align*}
3x - y &= 7 \\
2x + y &= 3 \\
\hline
5x &= 10 \\
\end{align*}
\]
Solve for x
\[ x = 2 \]

Plug \( x = 2 \) into either equation A or B
and solve for y

Equation B
\[
\begin{align*}
2(2) + y &= 3 \\
4 + y &= 3 \\
y &= -1
\end{align*}
\]
Answer: \((2, -1)\)

check:
\[
\begin{align*}
3(2) - (-1) &= 7 \\
2(2) + (-1) &= 3 \\
6 + 1 &= 7 \\
4 - 1 &= 3
\end{align*}
\]
Special Cases: No Solution or All points On The Line

Example 5

Equation A \[ 2x + 7y = 3 \]
Equation B \[ -2x - 7y = 5 \]

Add Equation A and Equation B

\[
\begin{align*}
2x + 7y &= 3 \\
-2x - 7y &= 5 \\
0 &= 8
\end{align*}
\]

Stop: Both the x and y terms canceled out and the remaining equation \( 0 = 8 \) is false.

The lines are parallel, they have no common points.

Answer: No Solution

Example 6

Equation A \[-4x - 5y = 3\]
Equation B \[ 4x + 5y = -3 \]

Add Equation A and Equation B

\[
\begin{align*}
-4x - 5y &= 3 \\
4x + 5y &= -3 \\
0 &= 0
\end{align*}
\]

Stop: Both the x and y terms canceled out and the remaining equation \( 0 = 0 \) is true.

Both equations describe the same line any point on \(-4x - 5y = 3\) would also be on \(4x + 5y = -3\)

Answer: All Points on \(4x + 5y = -3\) or

Answer: All Points on \(4x - 5y = 3\) either one of the above is correct.
What if adding Equation A and Equation B together does not eliminate one of the Variables?

The Multiplication Property of Equality

You can multiply both sides of an equation by the same number and still have an equivalent equation.

If the coefficients of the x or y terms are not the same number with different signs then the addition of the two equations will not eliminate one of the two variables. You can use the Multiplication Property of Equality to transform the two equations so that the coefficients of either the x or y terms add to zero. This step requires that you multiply one or both of the equations by a number that will cause either the x or y terms add to zero.

Example 7

Equation A \[ 4x + 3y = 16 \]
Equation B \[ -2x + y = 2 \]

multiply Equation B by 2 so that
Equation A has 4x and
Equation B has −4x

\[
\begin{align*}
4x + 3y &= 16 \\
-2x + y &= 2 \\
&
\end{align*}
\]

\[
\begin{align*}
4x + 3y &= 16 \\
-4x + 2y &= 4 \\
&
\end{align*}
\]

5y = 20
y = 4

Plug y = 4 into either equation A or B and solve for x

Equation A
4x + 3(4) = 16
4x + 12 = 16
4x = 4
x = 1
Answer: (1,4)

Example 8

Equation A \[ 4x + 3y = 2 \]
Equation B \[ 5x + 3y = 1 \]

multiply Equation A by −1 so that
Equation A has a −3y and
Equation B has a 3y

\[
\begin{align*}
-4x - 3y &= -2 \\
5x + 3y &= 1 \\
&
\end{align*}
\]

x = -1

Plug x = -1 into either equation A or B and solve for y

Equation A
4(-1) + 3y = 2
-4 + 3y = 2
3y = 6
y = 2
Answer: (-1,2)
Sometimes you can eliminate either the x terms or the y terms.
It’s your choice which one you decide to eliminate.

Example 9A shows a system where the **x terms are eliminated**
by multiplying Equation A by $-1$. This eliminates the x terms and
leaves an equation in terms of y.

**Example 9A**

Equation A $\begin{cases} -2x + 6y = 5 \\ -2x + 3y = 11 \end{cases}$

multiply Equation A by $-1$ so that
Equation A has 2x and
Equation B has $-2x$

\[
\begin{align*}
-1 \begin{cases} -2x + 6y = 5 \\ -2x + 3y = 11 \end{cases} \\
2x - 6y = -5 \\
-2x + 3y = 11
\end{align*}
\]

this will eliminate the x terms

Example 9B shows the **same system**
but the **y terms are eliminated**
by multiplying Equation B by $-2$. This eliminates the y terms and
leaves an equation in terms of x.

**Example 9B**

Equation A $\begin{cases} -2x + 6y = 5 \\ -2x + 3y = 11 \end{cases}$

multiply Equation B by $-2$ so that
Equation A has 6y and
Equation B has $-6y$

\[
\begin{align*}
\begin{cases} -2x + 6y = 5 \\ -2x + 3y = 11 \end{cases} \rightarrow \\
\begin{cases} -2x + 6y = 5 \\ 4x - 6y = -22 \end{cases}
\end{align*}
\]

this will eliminate the y terms
Example 10

Equation A \[ 4x + 3y = 16 \]
Equation B \[ -2x + y = 2 \]

Multiply Equation B by 2 to eliminate the x terms

\[
\begin{align*}
4x + 3y &= 16 \\
-2x + y &= 2
\end{align*}
\]

Multiply Equation B by 2

\[
\begin{align*}
4x + 3y &= 16 \\
-8x + 2y &= 4
\end{align*}
\]

Now add the two equations

\[
\begin{align*}
5y &= 20 \\
y &= 4
\end{align*}
\]

Plug \( y = 4 \) into either equation A or B and solve for x

Equation B

\[
\begin{align*}
4x + 3(4) &= 16 \\
4x + 12 &= 16 \\
4x &= 4 \\
x &= 1
\end{align*}
\]

Answer: \(( 1, 4)\)

check:

\[
\begin{align*}
4(1) + 3(4) &= 16 \\
-2(1) + (4) &= 2 \\
4 + 12 &= 16 \\
-2 + 4 &= 2
\end{align*}
\]

Example 11

Equation A \[ 3x + y = 2 \]
Equation B \[ 2x + 3y = 20 \]

Multiply Equation A by 3 to eliminate the y terms

\[
\begin{align*}
3x + y &= 2 \\
2x + 3y &= 20
\end{align*}
\]

Multiply Equation A by 3

\[
\begin{align*}
9x + 3y &= 6 \\
2x + 3y &= 20
\end{align*}
\]

Now add the two equations

\[
\begin{align*}
7x &= 14 \\
x &= 2
\end{align*}
\]

Plug \( x = -2 \) into either equation A or B and solve for y

Equation A

\[
\begin{align*}
3(-2) + y &= 2 \\
-6 + y &= 2 \\
y &= 8
\end{align*}
\]

Answer: \((-2, 8)\)

check:

\[
\begin{align*}
3(-2) + (8) &= 2 \\
2(-2) + 3(8) &= 20 \\
-6 + 8 &= 2 \\
-4 + 24 &= 20
\end{align*}
\]
Example 12

\[\begin{align*}
\text{Equation A} & : 2x + 2y = 7 \\
\text{Equation B} & : 4x - 3y = -7
\end{align*}\]

Multiply Equation A by \(-2\) to eliminate the \(x\) terms

\[-2\begin{align*}
2x + 2y &= 7 \\
4x - 3y &= -7
\end{align*}\]

\[-4x - 4y = -14
\]

Add the two equations

\[-7y = -21\]

Solve for \(y\)

\[y = 3\]

Plug \(y = 3\) into either equation A or B and solve for \(x\)

Equation B

\[\begin{align*}
4x - 3(3) &= -7 \\
4x - 9 &= -7 \\
4x &= 2 \\
x &= \frac{1}{2}
\end{align*}\]

Answer: \(\left(\frac{1}{2}, 3\right)\)

check:

\[\begin{align*}
2\left(\frac{1}{2}\right) + 2(3) &= 7 \\
4\left(\frac{1}{2}\right) - 3(3) &= -7
\end{align*}\]

Example 13

\[\begin{align*}
\text{Equation A} & : 4x - 6y = -2 \\
\text{Equation B} & : 2x + 3y = 3
\end{align*}\]

Multiply Equation B by \(2\) to eliminate the \(y\) terms

\[\begin{align*}
4x - 6y &= -2 \\
4x + 6y &= 6
\end{align*}\]

Add the two equations

\[8x = 4\]

Solve for \(x\)

\[x = \frac{4}{8} = \frac{1}{2}\]

Plug \(x = \frac{1}{2}\) into either equation A or B and solve for \(y\)

Equation A

\[\begin{align*}
4\left(\frac{1}{2}\right) - 6y &= -2 \\
2 - 6y &= -2 \\
-6y &= -4 \\
y &= \frac{-4}{-6} = \frac{2}{3}
\end{align*}\]

Answer: \(\left(\frac{1}{2}, \frac{2}{3}\right)\)

check:

\[\begin{align*}
4\left(\frac{1}{2}\right) - 6\left(\frac{2}{3}\right) &= -2 \\
2\left(\frac{1}{2}\right) + 3\left(\frac{2}{3}\right) &= 3
\end{align*}\]

\[\begin{align*}
4x - 6y &= -2 \\
2x + 3y &= 3
\end{align*}\]
You may need to multiply both equations by different numbers to eliminate one of the variables

Example 14A
We will eliminate the x variables

Equation A \[2x - 3y = 3\]
Equation B \[3x + 4y = 13\]

You must multiply both rows by different numbers to eliminate a variable

Multiply Equation A by −3
Multiply Equation B by 2
to eliminate the x terms

\[-3\begin{align*}
2x - 3y &= 3 \\
3x + 4y &= 13
\end{align*}\]
\[-6x + 9y = -9\]
\[\begin{align*}
6x + 8y &= 26 \\
17y &= 17 \\
y &= 1
\end{align*}\]

Plug \(y = 1\) into either equation A or B and solve for x

Equation B
3x + 4(1) = 13
3x + 4 = 13
3x = 9
x = 3
Answer: \((3,1)\)

check:
\[
\begin{align*}
2(3) - 3(1) &= 3 \\
3(3) + 4(1) &= 13
\end{align*}
\]

Example 14B
This is the same problem as Example 14A but we will eliminate the y variables

Equation A \[2x - 3y = 3\]
Equation B \[3x + 4y = 13\]

You must multiply both rows by different numbers to eliminate a variable

Multiply Equation A by 4
Multiply Equation B by 3
to eliminate the y terms

\[\begin{align*}
2x - 3y &= 3 \\
3x + 4y &= 13
\end{align*}\]
\[8x - 12y = 12\]
\[9x + 12y = 39\]
\[17x = 51\]
\[x = 3\]

Plug \(x = 3\) into either equation A or B and solve for y

Equation A
2(3) - 3y = 3
6 - 3y = 3
−3y = −3
y = 1
Answer: \((3,1)\)

check:
\[
\begin{align*}
2(3) - 3(1) &= 3 \\
3(3) + 4(1) &= 13
\end{align*}
\]
Example 15A
We will eliminate the x variables

Equation A \(-3x + 5y = 9\)
Equation B \(4x + 2y = 14\)

You must multiply both rows by different numbers to eliminate a variable

Multiply Equation A by 4
Multiply Equation B by 3
to eliminate the x terms

\[
\begin{align*}
4 \cdot (-3x + 5y) &= 36 \\
3 \cdot (4x + 2y) &= 30
\end{align*}
\]

\[
\begin{align*}
-12x + 20y &= 36 \\
12x + 6y &= 42
\end{align*}
\]
Now add the two equations

\[
26y = 78
\]
Solve for y

\[
y = 3
\]

Plug \(y = 3\) into either equation A or B and solve for x

Equation A
\(-3x + 5(3) = 9\)
\(-3x + 15 = 9\)
\(-3x = -6\)
x = 2
Answer: \((2,3)\)

cHECK:
\[
\begin{align*}
-3(2) + 5(3) &= 9 \\
4(2) + 2(3) &= 14
\end{align*}
\]

Example 15B
This is the same problem as Example 15B but we will eliminate the y variables

Equation A \(-3x + 5y = 9\)
Equation B \(4x + 2y = 14\)

You must multiply both rows by different numbers to eliminate a variable

Multiply Equation A by \(-2\)
Multiply Equation B by 5
to eliminate the y terms

\[
\begin{align*}
-2 \cdot (-3x + 5y) &= 6x - 10y = -18 \\
5 \cdot (4x + 2y) &= 20x + 10y = 70
\end{align*}
\]
Now add the two equations

\[
26x = 52
\]
Solve for x

\[
x = 2
\]

Plug \(y = 2\) into either equation A or B and solve for x

Equation A
\(-3(2) + 5y = 9\)
\(-6 + 5y = 9\)
\(5y = 15\)
y = 3
Answer: \((2,3)\)

cHECK:
\[
\begin{align*}
-3(2) + 5(3) &= 9 \\
4(2) + 2(3) &= 14
\end{align*}
\]

Note: Example 11 and 12 solved the exact same system two different ways. Example 11 eliminated the x variables first and example 12 solved the same system again by eliminating the y variables. You will never be asked to solve a system both ways as we did above. This was done so you could see it does not matter which variable you chose to eliminate.
What if the system has fractions in the equations?

Eliminate the fractions and get a system without fractions.

Systems with fractions can look overwhelming at first. The key to making these systems easier is to multiply each equation by the LCD (least common denominator) for the fractions in that equation. This will eliminate the denominators and give you a new system with out fractions. Solve this new system like the examples above.

Note: The examples below will not be completely solved. The steps to eliminate the fractions and get a new system without fractions will be shown. The remaining steps to solve the system are left to the student.

Example 1

Equation A \[
\frac{x}{3} - \frac{y}{6} = \frac{-2}{3}
\]

Equation B \[
\frac{-x}{2} - \frac{y}{4} = 1
\]

First multiply each equation by the LCD for that equation.
The resulting equations will not contain any fractions.

Multiply Equation A by 6
Multiply Equation B by 4

Equation A \[
6 \left(\frac{x}{3}\right) - 6 \left(\frac{y}{6}\right) = 6 \left(\frac{-2}{3}\right)
\]

Equation B \[
4 \left(\frac{-x}{2}\right) - 4 \left(\frac{y}{4}\right) = 4 \left(\frac{1}{1}\right) \cdot (1)
\]

Equation A \[
2x - y = -4
\]

Equation B \[
-2x - y = 4
\]

Now you can solve this system like the previous examples.

Example 17

Equation A \[
\frac{3x}{4} - \frac{y}{6} = \frac{-3}{8}
\]

Equation B \[
\frac{x}{6} + \frac{3y}{4} = 2
\]

First multiply each equation by the LCD for that equation.
The resulting equations will not contain any fractions.

Multiply Equation A by 24
Multiply Equation B by 12

Equation A \[
24 \left(\frac{3x}{4}\right) - 24 \left(\frac{y}{6}\right) = 24 \left(\frac{-3}{8}\right)
\]

Equation B \[
12 \left(\frac{x}{6}\right) + 12 \left(\frac{3y}{4}\right) = 12 \left(\frac{1}{1}\right) \cdot (2)
\]

Equation A \[
18x - 4y = -9
\]

Equation B \[
2x + 9y = 24
\]

Now you can solve this system like the previous examples.