Section 10 – 1:

Solving Quadratic Equations Using the Quadratic Formula

Quadratic Equations are equations that have an x^2 term as the highest power. They are also called Second Degree Equations.

The Standard Form of a Second Degree Equation

$$Ax^2 + Bx + C = 0$$

The number in front of the x^2 term is referred to with the letter A

The number in front of the x term is referred to with the letter B

The constant number is referred to with the letter C

Note: The value of A cannot be zero. If it is then the equation will not be a Quadratic Equation.

Solving Quadratic Equations

In Chapter Six equations of the form $Ax^2 + Bx + C = 0$ were solved by the use of the Zero Factor Rule. This rule was used to solve **equations** that have **products (factors)** that are **equal to zero.**

$(x - 3)(x + 4) = 0$ $\quad$ $(x + 2)(x - 2) = 0$ $\quad$ $(4x)(x - 6) = 0$ $\quad$ $x(5x + 2) = 0$

The use of the Zero Factor Rule required that **the second degree expression could be factored.** If the second degree expression **cannot be factored** or you do not want to solve by factoring then another method can be used to solve the equation. This method can be used to solve **ALL** second degree expression whether they can be factored or not.

The **Quadratic Formula** uses the values for A, B and C in the equation $Ax^2 + Bx + C = 0$ and substitutes them into the formula $x = \dfrac{-B \pm \sqrt{B^2 - 4AC}}{2A}$ to find the **two values for x** that are solutions to the Second Degree Equation $Ax^2 + Bx + C = 0$

The Quadratic Formula

If $Ax^2 + Bx + C = 0$ **then** $x = \dfrac{-B \pm \sqrt{B^2 - 4AC}}{2A}$

Note: The value of A cannot be zero.

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Steps to solve a second degree equation by the use of the Quadratic Formula

Step 1. Get the equation in the form $Ax^2 + Bx + C = 0$

Step 2. List the values for A, B and C

Step 3. Find the value of the discriminate $B^2 - 4AC$

Step 4. Put the values for –B, $B^2 - 4AC$ and 2A into the **Quadratic Formula.**

Step **5. Reduce the square root completely** and then reduce the remaining fraction if possible.

Step 1. Get the equation in the form $Ax^2 + Bx + C = 0$

Get all the terms on one side of the = sign in the correct order. We will chose to do this in a way that keeps the Ax^2 term positive. Many of the problems in this section will be set up in this form at the start of the problem. If not, move the terms to one side by addition or subtraction.

Example 1

$5x^2 + 4x = 9$

subtract 9 from both sides

$x^2 + 4x - 9 = 9 - 9$

$x^2 + 4x - 9 = 0$

Example 2

$5x^2 = 3x - 5$

subtract 3x from both sides
and add 5 to both sides

$x^2 - 3x + 5 = 3x - 5 - 3x + 5$

$x^2 - 3x + 5 = 0$

Example 3

$7x^2 + 6 = -9x$

add 9x to both sides

$7x^2 + 6 + 9x = -9x + 9x$

$x^2 + 9x + 6 = 0$

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Step 2. List the values for A, B and C in $\text{Ax}^2 + \text{Bx} + \text{C} = 0$

Example 1

$8x^2 - 7x + 9 = 0$

$A = 8 \quad B = -7 \quad C = 9$

Example 2

$3x^2 + 6x - 5 = 0$

$A = 3 \quad B = 6 \quad C = -5$

Example 3

$4x^2 - x + 3 = 0$

$A = 4 \quad B = -1 \quad C = 3$

Step 3. Find the value of the discriminate $\text{B}^2 - 4\text{AC}$

The value under the radical sign is $\text{B}^2 - 4(\text{A})(\text{C})$ and is called the **discriminant.** The value of $\text{B}^2 - 4(\text{A})(\text{C})$ determines the type of number the solution will be and the amount of work required to solve the quadratic formula for x.

Example 1

Find $B^2 - 4(A)(C)$

for $2x^2 - x - 10 = 0$

$A = 2 \quad B = -1 \quad C = -10$

$1 - 4(2)(-10) = 1 + 80 = 81$

Example 2

Find $B^2 - 4(A)(C)$

$x^2 - 5x + 2 = 0$

$A = 1 \quad B = -5 \quad C = 2$

Find $B^2 - 4(A)(C)$
$25 - 4(1)(2) = 25 - 8 = 17$

Example 3

Find $B^2 - 4(A)(C)$

for $4x^2 + 6x + 1 = 0$

$A = 4 \quad B = 6 \quad C = 1$

Find $B^2 - 4(A)(C)$
$36 - 4(4)(1) = 36 - 16 = 20$

Example 4

Find $B^2 - 4(A)(C)$

for $4x^2 - 5x + 2 = 0$

$A = 4 \quad B = -5 \quad C = 2$

Find $B^2 - 4(A)(C)$
$25 - 4(4)(2) = 25 - 32 = -7$

Step 4. Put the values for –B, $B^2 - 4AC$ and 2A into the Quadratic Formula.

Example 1

If $4x^2 - 6x + 1 = 0$

then

$A = 4 \quad B = -6$ and $B^2 - 4(A)(C) = 20$

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

$$x = \frac{-(-6) \pm \sqrt{20}}{2(4)}$$

$$x = \frac{6 \pm \sqrt{20}}{8}$$

Example 2

If $x^2 + 4x - 12 = 0$

then

$A = 1 \quad B = 4$ and $B^2 - 4(A)(C) = 64$

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

$$x = \frac{-(4) \pm \sqrt{64}}{2(1)}$$

$$x = \frac{-4 \pm \sqrt{64}}{2}$$

The Quadratic Formula

If $Ax^2 + Bx + C = 0$ **then** $x = \dfrac{-B \pm \sqrt{B^2 - 4AC}}{2A}$ **or** $\left(x = \dfrac{-B + \sqrt{B^2 - 4AC}}{2A} \text{ or } x = \dfrac{-B - \sqrt{B^2 - 4AC}}{2A} \right)$

Notice that there are **two values for x that are solutions**. The ± sign means that the term will be + in one version of the solution and – in the other. Each of the separate expressions gives a separate solution for x. It is common to start the solution process with a single expression with the ± sign and then separate the expression into the separate solutions as part of the reducing process.

Example 1

$$x = \frac{5 \pm \sqrt{17}}{2}$$

which can also be written as two separate solutions

$$x = \frac{5 + \sqrt{17}}{2} \text{ or } x = \frac{5 - \sqrt{17}}{2}$$

Example 2

$$x = \frac{4 \pm \sqrt{11}}{8}$$

which can also be written as two separate solutions

$$x = \frac{4 + \sqrt{11}}{8} \text{ or } x = \frac{4 - \sqrt{11}}{8}$$

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Putting It All Together

The value under the radical sign is expressed as $B^2 - 4(A)(C)$ and is called the **discriminant.** The value of $B^2 - 4(A)(C)$ determines the type of number the solution will be and the amount of work required to solve the quadratic formula for x.

The value for the discriminant $B^2 - 4(A)(C)$
can be any one of four possible types of numbers:

Case 1. When the discriminant $B^2 - 4(A)(C)$ is a **perfect square** then the solution will be **two rational numbers**. It will require several steps to find the two rational numbers. First replace the perfect square under the radical with its whole number value. Next break the equation with the ± sign into **two separate fraction**s, one with a + sign and one with a – sign. Then **simplify each fraction**.

The solution will look like $x = \frac{1}{2}$ or $x = \frac{3}{4}$

Case 2. When the discriminant $B^2 - 4(A)(C)$ is a square root that **cannot** be reduced because it does not have a perfect square factor then the quadratic formula cannot be reduced further.

The solution will look like $x = \frac{5 + \sqrt{7}}{3}$ or $x = \frac{5 - \sqrt{7}}{3}$

Case 3. When the discriminant $B^2 - 4(A)(C)$ is not a perfect square but the irrational number **can** be reduced to a radical with a smaller number under it because it has a perfect square factor then the radical must be reduced. That will leave you with an expression that may also be reduced. If **all the three integers** outside the radical have a common factor then reduce them by the common factor as the last step.

The solution will look like $x = \frac{1 + 2\sqrt{5}}{7}$ or $\frac{1 - 2\sqrt{5}}{7}$

Case 4. When the discriminant $B^2 - 4(A)(C)$ is a **negative numbe**r under the square root then the solution will not be a real number. Stop reducing the solution as soon as you see $b^2 - 4(a)(c)$ is negative and write NRN.

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$$x = \frac{-4 \pm \sqrt{-15}}{6} = \text{NRN}$$

Case 1: The discriminant is a Perfect Square

When the discriminant $B^2 - 4(A)(C)$ is a **perfect square** then the solution will be **two rational numbers**. It will require several steps to find the two rational numbers. First replace the perfect square under the radical with its whole number value. Next break the equation with the ± sign into **two separate fractions**, one with a + sign and one with a – sign. Then **simplify each fraction**.

Example 1

Solve for x

$x^2 + 5x + 4 = 0$

$A = 1 \quad B = 5 \quad C = 4$

Find $B^2 - 4(A)(C)$

$25 - 4(1)(4) = 25 - 16 = 9$

$x = \frac{-5 \pm \sqrt{9}}{2}$ reduce $\sqrt{9}$

$x = \frac{-5 \pm 3}{2}$ break into the + and − parts

$x = \frac{-5 + 3}{2} = \frac{-2}{2} = -1$

or

$x = \frac{-5 - 3}{2} = \frac{-8}{2} = -4$

$x = -1 \text{ or } -4$

Example 2

Solve for x

$x^2 + 4x - 12 = 0$

$A = 1 \quad B = 4 \quad C = -12$

Find $B^2 - 4(A)(C)$

$16 - 4(1)(-12) = 16 + 48 = 64$

$x = \frac{-4 \pm \sqrt{64}}{2}$ reduce $\sqrt{64}$

$x = \frac{-4 \pm 8}{2}$ break into the + and − parts

$x = \frac{-4 + 8}{2} = \frac{4}{2} = 2$

or

$x = \frac{-4 - 8}{2} = \frac{-12}{2} = -6$

$x = 2 \text{ or } -6$

In many quadratic equations the coefficient in front of the x^2 term (the value for A) will be a number other than 1. In the examples below the value for A is greater than 1.

$$x = \frac{-\text{B} \pm \sqrt{\text{B}^2 - 4\text{AC}}}{2\text{A}}$$

Example 3

Solve for x

$2x^2 - x - 10 = 0$

$A = 2 \quad B = -1 \quad C = -10$

Find $B^2 - 4(A)(C)$

$1 - 4(-2)(-10) = 1 + 80 = 81$

$x = \frac{-(-1) \pm \sqrt{81}}{2(2)}$

$x = \frac{1 \pm \sqrt{81}}{4}$ reduce $\sqrt{81}$

$x = \frac{1 \pm 9}{4}$ break into the $+$ and $-$ parts

$x = \frac{1 + 9}{2} = \frac{10}{4} = \frac{5}{2}$

or

$x = \frac{1 - 9}{4} = \frac{-8}{4} = -2$

$x = \frac{5}{2}$ or -2

Example 4

Solve for x

$3x^2 + 7x - 6 = 0$

$A = 3 \quad B = 7 \quad C = -6$

Find $B^2 - 4(A)(C)$

$49 - 4(3)(-6) = 49 + 72 = 121$

$x = \frac{-(7) \pm \sqrt{121}}{2(3)}$ reduce $\sqrt{121}$

$x = \frac{-7 \pm \sqrt{121}}{6}$ reduce $\sqrt{121}$

$x = \frac{-7 \pm 11}{6}$ break into the $+$ and $-$ parts

$x = \frac{-7 + 11}{6} = \frac{4}{6} = \frac{2}{3}$

or

$x = \frac{-7 - 11}{6} = \frac{-18}{6} = -3$

$x = \frac{2}{3}$ or -3

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In many quadratic equations the coefficient in front of the x term (the value for B) or the constant term (the value for C) will be 0. If **B = 0 or C = 0** then the Quadratic Equation is seldom used as a solution technique. It is much faster and easier to **solve for x by factoring**. Examples 5A and 6A show show the factoring method. The same problem is then solved by the use of the Quadratic Equation to show how the Quadratic Equation could be used to get the same solutions as the factoring technique.

Example 5A

What if B = 0

Solve for x

$4x^2 - 9 = 0$

Factor

$(2x - 3)\ (2x + 3) = 0$

$x = \frac{3}{2}$ or $x = \frac{-3}{2}$

Example 6A

What if C = 0

Solve for x

$6x^2 - 3x = 0$

Factor

$3x(2x - 1) = 0$

$x = 0$ or $x = \frac{1}{2}$

Example 5B

Example 6B

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Solve for x

$4x^2 - 9 = 0$

$A = 4 \quad B = 0 \quad C = -9$

Find $B^2 - 4(A)(C)$

$0 - 4(4)(-9) = 144$

$x = \dfrac{0 \pm \sqrt{144}}{8}$ reduce $\sqrt{144}$

$x = \dfrac{0 \pm 12}{8}$ break into the $+$ and $-$ parts

$x = \dfrac{0+12}{8} = \dfrac{12}{8} = \dfrac{3}{2}$

or

$x = \dfrac{0-12}{8} = \dfrac{-12}{8} = \dfrac{-3}{2}$

$x = \dfrac{3}{2}$ or $\dfrac{-3}{2}$

Solve for x

$6x^2 - 3x = 0$

$A = 6 \quad B = -3 \quad C = 0$

Find $B^2 - 4(A)(C)$

$9 - 4(6)(0) = 9 - 0 = 9$

$x = \dfrac{3 \pm \sqrt{9}}{12}$ reduce $\sqrt{9}$

$x = \dfrac{3 \pm 3}{12}$ break into the $+$ and $-$ parts

$x = \dfrac{3+3}{12} = \dfrac{6}{12} = \dfrac{1}{2}$

or

$x = \dfrac{3-3}{12} = \dfrac{0}{12} = 0$

$x = \dfrac{1}{2}$ or 0

Case 2 : The discriminant cannot be reduced at all

When the discriminant $B^2 - 4(A)(C)$ is a square root that **cannot** be reduced at all because it does not have a perfect square factor then the quadratic formula cannot be reduced further.

Example 7

Example 8

Solve for x

$x^2 - 5x + 2 = 0$

$A = 1 \quad B = -5 \quad C = 2$

Find $B^2 - 4(A)(C)$

$25 - 4(1)(2) = 25 - 8 = 17$

$$x = \frac{5 \pm \sqrt{17}}{2}$$

The $\sqrt{17}$ cannot be reduced at all.
$\sqrt{17}$ does not have a perfect square factor or a pair of factors.
When this happens your answer is complete

$$x = \frac{5 \pm \sqrt{17}}{2}$$

which can also be written

$$x = \frac{5 + \sqrt{17}}{2} \text{ or } x = \frac{5 - \sqrt{17}}{2}$$

Solve for x

$4x^2 - 3x - 2 = 0$

$A = 4 \quad B = -5 \quad C = 2$

Find $B^2 - 4(A)(C)$

$9 - 4(3)(-2) = 9 + 24 = 33$

$$x = \frac{3 \pm \sqrt{33}}{8}$$

The $\sqrt{33}$ cannot be reduced at all.
$\sqrt{33}$ does not have a perfect square factor or a pair of factors. When this happens your answer is complete

$$x = \frac{3 \pm \sqrt{33}}{8}$$

which can also be written

$$x = \frac{3 + \sqrt{33}}{8} \text{ or } x = \frac{3 - \sqrt{33}}{8}$$

Case 3. **The discriminant** is not a perfect square but the irrational number **can** be reduced

When the discriminant $B^2 - 4(A)(C)$ is not a perfect square but the irrational number **can** be reduced to a radical with a smaller number under it because it has a perfect square factor then the radical must be reduced. That will leave you with an expression that may also be reduced. If **all the three integers** outside the radical have a common factor then reduce them by the common factor as the last step.

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Example 9

Solve for x

$4x^2 - 6x + 1 = 0$

$A = 4 \quad B = -6 \quad C = 1$

Find $B^2 - 4(A)(C)$

$36 - 4(4)(1) = 36 - 16 = 20$

$x = \dfrac{6 \pm \sqrt{20}}{8}$ reduce $\sqrt{20} = \sqrt{4 \bullet 5}$

$x = \dfrac{6 \pm 2\sqrt{5}}{8}$

all of the three integers outside the radical have a common factor

$x = \dfrac{\not{6}^3 \pm \not{2}^1\sqrt{5}}{\not{8}^4}$

$x = \dfrac{3 \pm 1\sqrt{5}}{4}$ break into the + and − parts

$x = \dfrac{3 + \sqrt{5}}{4}$ or $\dfrac{3 - \sqrt{5}}{4}$

Example 10

Solve for x

$x^2 + 4x - 2 = 0$

$A = 1 \quad B = 4 \quad C = -2$

Find $B^2 - 4(A)(C)$

$16 - 4(1)(-2) = 16 + 8 = 24$

$x = \dfrac{-4 \pm \sqrt{24}}{2}$ reduce $\sqrt{24} = \sqrt{4 \bullet 6}$

$x = \dfrac{-4 \pm 2\sqrt{6}}{2}$

all of the three integers outside the radical have a common factor

$x = \dfrac{-\not{4}^2 \pm \not{2}^1\sqrt{6}}{\not{2}^1}$

$x = \dfrac{-2 \pm 1\sqrt{6}}{1}$ break into the + and − parts

$x = -2 + \sqrt{6}$ or $-2 - \sqrt{6}$

Case 4: The discriminant $B^2 - 4(A)(C)$ is a **negative numbe**r

When the discriminant $B^2 - 4(A)(C)$ is a **negative numbe**r under the square root then the solution will not be a real number. Stop reducing the solution as soon as you see $b^2 - 4(a)(c)$ is negative and write NRN.

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Example 11

Solve for x

$6x^2 - 4x + 1 = 0$

$A = 6 \quad B = -4 \quad C = 1$

Find $B^2 - 4(A)(C)$

$16 - 4(6)(1) = 16 - 24 = -8$

If the final number under the radical(the discriminant) reduces to a negitive number then NO REAL NUMBERS will work and we write

$x = \text{NRN}$

Example 12

Solve for x

$5x^2 + 2x + 3 = 0$

$A = 5 \quad B = 2 \quad C = 3$

Find $B^2 - 4(A)(C)$

$4 - 4(5)(3) = 4 - 60 = -54$

If the final number under the radical(the discriminant) reduces to a negitive number then NO REAL NUMBERS will work and we write

$x = \text{NRN}$

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