

Chapter 0 Review of Arithmetic

Fractions

There are several terms that are commonly used when working with fractions.

Fraction: The ratio of two numbers. We use a division bar to show this ratio. The number on **the top** of the fraction is called the **Numerator** and the number on **the bottom** of the fraction is called the **Denominator**. The fraction bar indicates division. Since division by 0 is undefined, a fraction with a denominator of 0 is undefined.

$$\frac{4}{5} \text{ is a fraction and } \frac{4}{5} \begin{array}{l} \leftarrow \text{the numerator is 4} \\ \leftarrow \text{the denominator is 5} \end{array}$$

A **proper fraction** is a fraction where the **numerator is smaller** than the denominator.

An **improper fraction** is a fraction where the **numerator is larger** than the denominator.

$$\frac{4}{5} \text{ is a } \mathbf{proper} \text{ fraction} \qquad \frac{14}{5} \text{ is an } \mathbf{improper} \text{ fraction}$$

A Fraction Reduced to Lowest Terms

A **fraction is considered to be reduced to lowest terms** if there is no number other than 1 that can divide into **BOTH** the **Numerator** and the **Denominator**.

$$\frac{7}{12} \text{ is a fraction in lowest terms}$$

$$\frac{18}{14} \text{ is not a fraction in lowest terms because } 2 \text{ will divide into both the 18 and the 14}$$

$$\frac{15}{8} \text{ is a fraction in lowest terms}$$

$$\frac{14}{21} \text{ is not a fraction in lowest terms because } 7 \text{ will divide into both the 14 and the 21}$$

The process of changing a fraction that is not in lowest terms into a fraction that is in lowest terms is called **reducing the fraction to lowest terms**.

Reducing a Fraction to Lowest Terms

1. Find the **largest number** that will divide evenly into **both** the Numerator and the Denominator.
This number is called the **greatest common factor** (GCF) of the numerator and denominator.
2. Divide the greatest common factor into both the numerator and denominator.

Example 1

a 3 will divide into
both 21 and 18

$$\frac{21}{18} = \frac{21^3}{18^3} = \frac{7}{6}$$

Example 2

a 9 will divide into
both 27 and 36

$$\frac{27}{36} = \frac{27^3}{36^3} = 4$$

Example 3

a 7 will divide into
both 42 and 49

$$\frac{42}{49} = \frac{42^7}{49^7} = \frac{6}{7}$$

Mixed Numbers

A **mixed number** is a whole number with a **proper fraction** following it. The proper fraction part of the mixed number must be reduced to lowest terms.

$3\frac{4}{5}$ is a mixed number and is read three and four fifths.

$5\frac{3}{4}$ is a mixed number and is read five and three fourths.

$6\frac{7}{5}$ is not a correct mixed number because the fraction part $\frac{7}{5}$ is not a proper fraction

$8\frac{2}{4}$ is not a correct mixed number because the proper fraction part $\frac{2}{4}$ is not reduced.

Converting an Improper Fraction into a Mixed Number

1. **Reduce the improper fraction** to lowest terms if that is possible.
2. Divide the denominator into the numerator to get a whole number and a remainder.
3. The whole number of times the denominator goes into the numerator becomes the whole number part of the mixed number. The remainder from the division problem is put over the denominator and becomes the fraction part of the mixed number.

Example 1

To convert $\frac{17}{5}$ into a mixed number divide 5 into 17.
$$\begin{array}{r} 3 \text{ r} 2 \\ 5 \overline{)17} \end{array}$$
 or 3 remainder 2.

$$\text{So } \frac{17}{5} = 3 \frac{2}{5}$$

Example 2

Convert $\frac{28}{8}$ into a mixed number by reducing $\frac{28}{8}$.
$$\frac{28}{8} = \frac{28^7}{8^2} = \frac{7}{2}$$
 Now convert $\frac{7}{2}$ into a

mixed number by dividing 2 into 7.
$$\begin{array}{r} 3 \text{ r} 1 \\ 2 \overline{)7} \end{array}$$
 or 3 remainder 1.

$$\text{So } \frac{7}{2} = 3 \frac{1}{2}$$

Example 3

Convert $\frac{28}{6}$ into a mixed number by reducing $\frac{28}{6}$.
$$\frac{28^{14}}{6^3} = \frac{14}{3}$$

Now convert $\frac{14}{3}$ into a mixed number by dividing 3 into 14.
$$\begin{array}{r} 4 \text{ r} 2 \\ 3 \overline{)14} \end{array}$$
 or 4 remainder 2.

$$\text{So } \frac{14}{3} = 4 \frac{2}{3}$$

Converting a Mixed Number into an Improper Fraction.

1. Multiply the whole number in front of the fraction by the denominator of the fraction and **add that product** to the numerator. This number becomes the numerator of the answer.
2. The denominator for the answer is the denominator of the original fraction.

Example 1

$$2\frac{5}{6} = \frac{(2 \cdot 6) + 5}{6} = \frac{17}{6}$$

Example 2

$$5\frac{3}{4} = \frac{(5 \cdot 4) + 3}{4} = \frac{23}{4}$$

Example 3

$$1\frac{4}{5} = \frac{(1 \cdot 5) + 4}{5} = \frac{9}{5}$$

Multiplying Fractions

When you multiply two (or more) fractions together the numerator of the answer is found by multiplying all the tops (numerators) of the fractions together and the denominator is found by multiplying all the bottoms (denominators) of the fractions together. **The resulting answer must be reduced if possible.**

Example 1

$$\begin{aligned} & \frac{2}{3} \cdot \frac{4}{5} \\ &= \frac{2 \cdot 4}{3 \cdot 5} \\ &= \frac{8}{15} \end{aligned}$$

Example 2

$$\begin{aligned} & \frac{3}{5} \cdot \frac{2}{4} = \frac{3 \cdot 2}{5 \cdot 4} \\ &= \frac{6}{20} \\ &= \frac{6^3}{20^{10}} \\ &= \frac{3}{10} \end{aligned}$$

Example 3

$$\begin{aligned} & \frac{5}{6} \cdot \frac{3}{4} = \frac{5 \cdot 3}{6 \cdot 4} \\ &= \frac{15}{24} \\ &= \frac{15^5}{24^8} \\ &= \frac{5}{8} \end{aligned}$$

Save Time: **Reduce First** , Then Multiply

You will save time if you start by trying to reduce the fractions before you multiply. You can cancel out any common factor in the denominator with any any common factor in the numerator.

Example 4

$$\begin{aligned} & \frac{6}{5} \cdot \frac{3}{8} \\ &= \frac{6^3}{5} \cdot \frac{3}{8^4} \\ &= \frac{9}{20} \end{aligned}$$

Example 5

$$\begin{aligned} & \frac{6}{5} \cdot \frac{3}{8} \\ &= \frac{6^3}{5} \cdot \frac{3}{8^4} \\ &= \frac{9}{20} \end{aligned}$$

Note: If the answer is an improper fraction you may be asked to convert it to a mixed number. Be sure to read the instructions.

Example 6

$$\begin{aligned} & \frac{21}{3} \cdot \frac{5}{6} \\ &= \frac{21^1}{3} \cdot \frac{5}{6^1} \\ &= \frac{35}{6} = 5\frac{5}{6} \end{aligned}$$

Example 7

$$\begin{aligned} & \frac{15}{4} \cdot \frac{8}{9} \\ &= \frac{15^1}{4^1} \cdot \frac{8^2}{9^2} \\ &= \frac{10}{3} = 3\frac{1}{3} \end{aligned}$$

Multiplying More Than Two Fractions

You multiply more than two fractions in the same manner as above.

Example 8

$$\begin{aligned} & \frac{1}{2} \cdot \frac{4}{5} \cdot \frac{10}{3} \\ &= \frac{1}{2^1} \cdot \frac{4^2}{5^1} \cdot \frac{10^2}{3} \\ &= \frac{4}{3} = 1\frac{1}{3} \end{aligned}$$

Example 9

$$\begin{aligned} & \frac{1}{8} \cdot \frac{5}{7} \cdot 12 \\ &= \frac{1}{8^2} \cdot \frac{5}{7} \cdot \frac{12^3}{1} \\ &= \frac{15}{14} = 1\frac{1}{14} \end{aligned}$$

Multiplying Mixed Numbers

Note: **To multiply mixed numbers you MUST first convert each mixed number into an improper fraction** and then multiply the improper fractions.

Example 9

$$\begin{aligned} & 1\frac{2}{5} \cdot 2\frac{1}{4} \\ &= \frac{7}{5} \cdot \frac{9}{4} \\ &= \frac{63}{20} \\ &= 3\frac{3}{20} \end{aligned}$$

Example 10

$$\begin{aligned} & 1\frac{2}{3} \cdot 2\frac{1}{5} \\ &= \frac{5}{3} \cdot \frac{11}{5} \\ &= \frac{5^1}{3} \cdot \frac{11}{5^1} \\ &= \frac{11}{3} = 3\frac{2}{3} \end{aligned}$$

Example 11

$$\begin{aligned} & 3\frac{1}{2} \cdot 3\frac{1}{3} \\ &= \frac{7}{2} \cdot \frac{10}{3} \\ &= \frac{7}{2^1} \cdot \frac{10^5}{3} \\ &= \frac{35}{3} = 11\frac{2}{3} \end{aligned}$$

Division with Fractions

To divide two fractions you **invert the fraction to the right** of the division sign (switch places with the denominator and numerator) and then **change the division sign to a multiplication sign**. You now have a multiplication problem. Multiply the fractions as you did in the previous problems.

Example 1

$$\frac{5}{14} \div \frac{10}{7}$$

Flip the fraction
on the right

$$= \frac{5}{14} \cdot \frac{7}{10}$$

multiply

$$= \frac{5^1}{14^2} \cdot \frac{7^1}{10^2}$$

$$= \frac{1}{4}$$

Example 2

$$\frac{5}{6} \div \frac{7}{10}$$

Flip the fraction
on the right

$$= \frac{5}{6} \cdot \frac{10}{7}$$

multiply

$$= \frac{5}{6^3} \cdot \frac{10^5}{7}$$

$$= \frac{25}{21} = 1\frac{4}{21}$$

Example 3

$$\frac{35}{2} \div 15$$

Flip the fraction
on the right

$$= \frac{35}{2} \cdot \frac{1}{15}$$

multiply

$$= \frac{35^7}{2} \cdot \frac{1}{15^3}$$

$$= \frac{7}{6} = 1\frac{1}{6}$$

Dividing Mixed Numbers

Note: **To divide mixed numbers** you **MUST** first convert each mixed number into an improper fraction, then invert the second fraction and then multiply the fractions.

Example 4

$$2\frac{1}{2} \div 3\frac{1}{3} \text{ (convert to improper fractions)}$$

$$= \frac{5}{2} \div \frac{10}{3} \text{ (invert the second fraction)}$$

$$= \frac{5}{2} \cdot \frac{3}{10} \text{ (multiply the fractions)}$$

$$= \frac{5^1}{2} \cdot \frac{3}{10^2}$$

$$= \frac{3}{4}$$

Example 5

$$1\frac{5}{9} \div 9\frac{1}{3} \text{ (convert to improper fractions)}$$

$$= \frac{14}{9} \div \frac{28}{3} \text{ (invert the second fraction)}$$

$$= \frac{14}{9} \cdot \frac{3}{28} \text{ (multiply the fractions)}$$

$$= \frac{14^1}{9^3} \cdot \frac{3^1}{28^2}$$

$$= \frac{1}{6}$$

Adding and Subtracting Fractions with a Common Denominator

To add or subtract fractions, each fraction **must have the same denominator**. We call the number that must be the same for each denominator a **common denominator**. To add or subtract a fraction with common denominators add or subtract the numerators and put that answer over the common denominator. Remember to reduce the fraction to lowest terms if possible.

Example 1

$$\begin{aligned} & \frac{3}{5} + \frac{1}{5} \\ &= \frac{3+1}{5} \\ &= \frac{4}{5} \end{aligned}$$

Example 2

$$\begin{aligned} & \frac{7}{8} - \frac{5}{8} \\ &= \frac{7-5}{8} \\ &= \frac{2}{8} = \frac{1}{4} \end{aligned}$$

Example 3

$$\begin{aligned} & \frac{6}{8} + \frac{7}{8} + \frac{1}{8} \\ &= \frac{6+7+1}{8} \\ &= \frac{14}{8} = \frac{7}{4} = 1\frac{3}{4} \end{aligned}$$

Adding and Subtracting Mixed Numbers with a Common Denominator

It is common to write the addition and subtraction of **mixed numbers** in a vertical format with one fraction written under the other. You **DO NOT** need to convert each mixed number to an improper fraction to add or subtract the fractions.

Example 4

$$\begin{array}{r} 3\frac{1}{5} \\ + 2\frac{3}{5} \\ \hline 5\frac{4}{5} \end{array}$$

Example 5

$$\begin{array}{r} 2\frac{4}{9} \\ + 5\frac{2}{9} \\ \hline 7\frac{6}{9} = 7\frac{2}{3} \end{array}$$

Example 6

$$\begin{array}{r} 6\frac{7}{9} \\ - 1\frac{4}{9} \\ \hline 5\frac{3}{9} = 5\frac{1}{3} \end{array}$$

Example 7

$$\begin{array}{r} 3\frac{7}{8} \\ - 1\frac{1}{8} \\ \hline 2\frac{6}{8} = 2\frac{3}{4} \end{array}$$

When you add mixed numbers the fraction part of the answer may be an improper fraction. If this happens you **must reduce the improper fraction to a mixed number** and then add the two whole numbers to get a final answer.

Example 8

$$\begin{array}{r} 2\frac{4}{5} \text{ (add the fractions and)} \\ + 3\frac{3}{5} \text{ (add the whole numbers)} \\ \hline = 5 + \frac{7}{5} \\ = 5 + 1\frac{2}{5} \\ = 6\frac{2}{5} \end{array}$$

Example 9

$$\begin{array}{r} 7\frac{5}{6} \text{ (add the fractions and)} \\ + 1\frac{5}{6} \text{ (add the whole numbers)} \\ \hline = 8 + \frac{10}{6} \\ = 8 + 1\frac{4}{6} \\ = 9\frac{4}{6} = 9\frac{2}{3} \end{array}$$

Subtracting Fractions by Borrowing

When you subtract two fractions the bottom fraction is subtracted from the top fraction. This requires **the top fraction to be larger than the bottom fraction**. If this is not the case then you **must make the top fraction larger by borrowing a 1 from the whole number next to it**. After you borrow the one you must convert the top fraction to an improper fraction. At this point the top fraction will be larger than the bottom one and you can subtract the fractions.

Example 1

$$\begin{array}{r} 9\frac{1}{5} = 9^8 + 1\frac{1}{5} = 8\frac{6}{5} \\ - 2\frac{4}{5} = \quad - 2\frac{4}{5} \\ \hline \quad \quad \quad 6\frac{2}{5} \end{array}$$

Example 2

$$\begin{array}{r} 3\frac{2}{7} = 3^2 + 1\frac{2}{7} = 2\frac{9}{7} \\ - 1\frac{5}{7} = \quad - 1\frac{5}{7} \\ \hline \quad \quad \quad 1\frac{4}{7} \end{array}$$

Example 3

$$\begin{array}{r} 9 = 9^8 + 1 = 8\frac{3}{3} \\ - 6\frac{2}{3} = \quad - 6\frac{2}{3} \\ \hline \quad \quad \quad 2\frac{2}{3} \end{array}$$

Example 4

$$\begin{array}{r} 5 = 5^4 + 1 = 4\frac{6}{6} \\ - 4\frac{5}{6} = \quad - 4\frac{5}{6} \\ \hline \quad \quad \quad 1\frac{1}{6} \end{array}$$

Adding and Subtracting Fractions without a Common Denominator

To add or subtract fractions, each fraction **MUST** have the same denominator. If the fractions do not have a common denominator you must make new fractions that both have a common denominator. The Least Common Denominator (LCD) is the smallest number all of the denominators divide into. To make each fraction's denominator into the LCD, multiply the top and bottom of each fraction by the number that will cause each denominator to become the LCD.

Example 1 The LCD is 12

$$\begin{array}{r} \frac{1}{6} = \frac{1 \cdot 2}{6 \cdot 2} = \frac{2}{12} \\ + \frac{1}{4} = \frac{3 \cdot 1}{3 \cdot 4} = + \frac{3}{12} \\ \hline = \frac{5}{12} \end{array}$$

Example 2 The LCD is 12

$$\begin{array}{r} \frac{2}{3} = \frac{2 \cdot 4}{3 \cdot 4} = \frac{8}{12} \\ + \frac{3}{4} = \frac{3 \cdot 3}{4 \cdot 3} = + \frac{9}{12} \\ \hline = \frac{17}{12} \\ = 1 \frac{5}{12} \end{array}$$

Example 3 The LCD is 15

$$\begin{array}{r} \frac{3}{5} = \frac{3 \cdot 3}{5 \cdot 3} = \frac{9}{15} \\ + \frac{2}{3} = \frac{2 \cdot 5}{3 \cdot 5} = + \frac{10}{15} \\ \hline = \frac{19}{15} \\ = 1 \frac{4}{15} \end{array}$$

Example 4 The LCD is 6

$$\begin{array}{r} 2 \frac{1}{3} = \frac{1 \cdot 2}{3 \cdot 2} = 2 \frac{2}{6} \\ + 1 \frac{1}{2} = \frac{1 \cdot 3}{2 \cdot 3} = + 1 \frac{3}{6} \\ \hline = 3 \frac{5}{6} \end{array}$$

Example 5 The LCD is 12

$$\begin{array}{r} 5 \frac{3}{4} = \frac{3 \cdot 2}{4 \cdot 2} = 5 \frac{6}{8} \\ + 3 \frac{3}{8} = \frac{3 \cdot 1}{8 \cdot 1} = + 3 \frac{3}{8} \\ \hline = 8 \frac{9}{8} \\ = 9 \frac{1}{8} \end{array}$$

Example 6 The LCD is 12

$$\begin{array}{r} \frac{3}{4} = \frac{3 \cdot 3}{4 \cdot 3} = \frac{9}{12} \\ - \frac{2}{3} = \frac{2 \cdot 4}{3 \cdot 4} = - \frac{8}{12} \\ \hline = \frac{1}{12} \end{array}$$

Example 7 The LCD is 15

$$\begin{array}{r} \frac{4}{5} = \frac{4 \cdot 3}{5 \cdot 3} = \frac{12}{15} \\ - \frac{1}{3} = \frac{1 \cdot 5}{3 \cdot 5} = - \frac{5}{15} \\ \hline = \frac{7}{15} \end{array}$$

Example 8 The LCD is 24

$$\begin{array}{r} 4 \frac{5}{6} = \frac{5 \cdot 4}{6 \cdot 4} = 4 \frac{20}{24} \\ - 2 \frac{3}{8} = \frac{3 \cdot 3}{8 \cdot 3} = - 2 \frac{9}{24} \\ \hline = 2 \frac{11}{24} \end{array}$$

Example 9 The LCD is 18

$$\begin{array}{r} 7 \frac{5}{6} = \frac{5 \cdot 3}{6 \cdot 3} = 7 \frac{15}{18} \\ - 1 \frac{4}{9} = \frac{4 \cdot 2}{9 \cdot 2} = - 1 \frac{8}{18} \\ \hline = 6 \frac{7}{18} \end{array}$$

Converting A Fraction To A Decimal Number

To convert a fraction to its decimal form **divide the denominator into the numerator**. If you have a mixed number you can change it into an improper fraction first if you like.

Example 1

Convert $\frac{2}{5}$ to a decimal

$$5 \overline{)2.0} = 5 \overline{)2.0}$$

$$= 0.4$$

Example 2

Convert $\frac{3}{4}$ to a decimal

$$4 \overline{)3.0} = 4 \overline{)3.00}$$

$$= 0.75$$

Example 3

Convert $2\frac{1}{2}$ to a decimal

$$2\frac{1}{2} = \frac{5}{2}$$

$$2 \overline{)5.0} = 2 \overline{)5.0}$$

$$= 2.5$$

Converting A Decimal Number To A Fraction

Every decimal number can be converted into a proper fraction or a mixed number. The digits to the **LEFT** of the decimal are the whole number part of the mixed number. The digits to the **RIGHT** of the decimal represent the fractional part. The decimal place of the digit farthest to the right determines the value of the denominator of the fraction.

Example 1

Convert .9 into a fraction

The first place to the right of a decimal is the **tenths place**.

0.9 is read 9 tenths and is written as the fraction $\frac{9}{10}$

5.7 is read 5 and 7 tenths and is written as the mixed number $5\frac{7}{10}$

Example 2

Convert .18 into a fraction

The second place to the right of the decimal is the **hundredth place**.

.18 is read 18 hundredths and is written as the fraction $\frac{18}{100}$

4.27 is read 4 and 27 hundredths and is written as the mixed number $4\frac{27}{100}$

To convert a decimal with number to the left of the decimal place the whole number part in front of a fraction. Then convert the number to the right of the decimal to a fraction and reduce. The denominator of the fraction will be 10 or 100 or 1000 etc. depending on where the last digit ends. **You must then reduce the fraction if it can be reduced.**

Example 3

.4
is read 4 tenths

$$= \frac{4}{10}$$

$$= \frac{2}{5}$$

Example 4

0.75
is read 75 hundredths

$$= \frac{75}{100}$$

$$= \frac{75^3}{100^4} = \frac{3}{4}$$

Example 5

2.35
is read 2 and 35 hundredths

$$= 2 \frac{35}{100}$$

$$= 2 \frac{35^7}{100^{20}}$$

$$= 2 \frac{7}{20}$$

Converting A Percent To A Decimal

To convert a percent (part of a hundred) to its decimal form **move the decimal two places to the left** and remove the % sign.

$$23\% = .23$$

$$54\% = .54$$

$$345\% = 3.45$$

$$43\% = .43$$

If there are not enough digits to allow you to move the decimal to the left two places then add zeros to the left of your number and then move the decimal two places to the left.

$$8.4\% = 08.4\% = .084$$

$$.72\% = 00.72\% = .0072$$

$$.023\% = 00.023\% = .00023$$

Converting A Decimal Number To A Percent.

To convert a decimal number to its percent form move the decimal two places to the right and add a % sign.

$$.34 = 34\%$$

$$3.56 = 356\%$$

$$0.03 = 3\%$$

$$0.0023 = .23\%$$

Basic Percent Problems

There are 3 types of percent problems that can be solved by the use of one step equations.

What number is 24% of 30 What percent of 32 is 8? 12 is 25% of what number?

The method we will use requires that we translate each problem into a one step algebra equation and then solve that equation for the unknown. We first note that any percent in the problem will be converted into it's decimal form. The key to translating each word problem into an equation is based on changing the unknown into a variable like n or x.

The word **is** translates to an = sign. The word **of** translates to **multiply**. The phrase "What percent" and "what number" translates to x.

English		Algebra	English		Algebra
of	translates to	•	what number	translates to	x
is	translates to	=	what percent	translates to	x

Solving Percent Problems using equations

- Step 1: Translate the english sentence into an algebra equation
- Step 2. Solve for x. if x equals a fraction that can be reduced then reduce the fraction
- Step 3. Perform the long division or multiplication showing the work.
- Step 4. Be sure to write your final answer in problems that ask for percent with a % sign.

Example 1

What number is 24% of 32

$$\begin{array}{ccccccc}
 \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \\
 x & = & .24 & \cdot & 32 & &
 \end{array}$$

$$x = .24 \cdot 32$$

multiply 32 by .24

$$\begin{array}{r}
 32 \\
 \times .24 \\
 \hline
 128 \\
 + 640 \\
 \hline
 768 \\
 x = 7.68
 \end{array}$$

Example 2

What percent of 32 is 24

$$\begin{array}{ccccccc}
 \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \\
 x & \cdot & 32 & = & 24 & &
 \end{array}$$

$$\frac{x \cdot 32}{32} = \frac{24}{32}$$

$$x = \frac{24}{32} = \frac{3}{4}$$

divide 3 by 4

$$\begin{array}{r}
 .75 \\
 4 \overline{) 3.00} \\
 \underline{-28} \\
 20 \\
 \underline{-20} \\
 0 \\
 x = 75\%
 \end{array}$$

Example 3

14 is 25% of what number

$$\begin{array}{ccccccc}
 \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \\
 14 & = & .25 & \cdot & x & &
 \end{array}$$

$$\frac{14}{.25} = \frac{.25 \cdot x}{.25}$$

$$\frac{14}{.25} = x$$

divide 14 by .25

$$\begin{array}{r}
 56 \\
 25 \overline{) 1400} \\
 \underline{-125} \\
 150 \\
 \underline{-150} \\
 0 \\
 x = 56
 \end{array}$$